# Domains of Bosonic Functional Integrals and Some Applications to the Mathematical Physics of Path Integrals and String Theory

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**Abstract**

By means of the Minlos Theorem on support of cylindrical measures on vectorial topological spaces, we present several results on the rigorous definitions of Euclidean path integrals and applications to some problems on non-linear diffusion, nonlinear wave propagations and covariant Polyakov's path Integrals yielding news results on the subject as well.

## 1 Introduction

Since the result of R.P. Feynman on representing the initial value solution of Schrodinger Equation by means of an analytically time continued integration on a infinite - dimensional space of functions, the subject of Euclidean Functional Integrals representations for Quantum Systems has became the mathematical - operational framework to analyze Quantum Phenomena and stochastic systems as showed in the previous decades of research on Theoretical Physics ([1]–[3]).

One of the most important open problem in the mathematical theory of Euclidean Functional Integrals is that related to implementation of sound mathematical approxima-

tions to these Infinite-Dimensional Integrals by means of Finite-Dimensional approximations outside of the always used [computer oriented] Space-Time Lattice approximations (see [2], [3] - chap. 9). As a first step to tackle upon the above cited problem it will be needed to characterize mathematically the Functional Domain where these Functional Integrals are defined.

The purpose of this paper is to present in section II, the formulation of Euclidean Quantum Field theories as Functional Fourier Transforms by means of the Bochner-Martin-Kolmogorov theorem for Topological Vector Spaces ([4], [5] - theorem 4.35) and suitable to define and analyze rigorously Functional Integrals by means of the well-known Minlos theorem ([5] - theorem 4.312 and [6] - part 2) and presented in full details in section 3.

In section 4, we present news results on the difficult problem of defining rigorously infinite-dimensional quantum field path integrals in general space times $\Omega \subset R^\nu$ $(\nu = 2, 4, \dots)$ by means of the analytical regularization scheme.

In section 3, we present a framework to write equilibrium measures for some non-linear diffusion problems. In the short section 4 we show the usefulness of our results on domain of path integrals by analyzing the problem of the support of generalized white - noise process with different strenghts. In section 5, we present the Ergodic Theorem applied to some problem in non-linear wave propagations through path integrals. In section 6 and 7 we review the A.M. Polyakov path integral analysis of String theory with corrections and improvements by considering a carefull analysis of the correct functional integrals variables on the underlying theory. Finally we call the reader attention that the appendices $A$ and $B$ has as important new results as these showed in the bulk of this work.

## 2 The Euclidean Schwinger Generating Functional as a Functional Fourier Transform

The basic object in a scalar Euclidean Quantum Field Theory in $R^D$ is the Schwinger Generating Functional (see refs. [1], [3]).

$$Z[j(x)] = \langle \Omega_{VAC} | \exp \left( i \int d^D x j(\vec{x}, it) \phi^{(m)}(\vec{x}, it) \right) | \Omega_{VAC} \rangle \tag{1}$$

where $\phi^{(m)}(\vec{x}, it)$ is the supposed Self-Adjoint Minkowski Quantum Field analytically continued to imaginary time and $j(x) = j(\vec{x}, it)$ is a set of functions belonging to a given Topological Vector Space of functions denoted by $E$ which topology is not specified yet and will be called the Schwinger Classical field source space. It is important to remark that $\{\phi^m(\vec{x}, it)\}$ is a commuting Algebra of Self-Adjoints operators as Symanzik has pointed out ([7]).

In order to write eq.(1) as an Integral over the space $E^{alg}$ of all linear functionals on the Schwinger Source Space $E$ (the called Algebraic Dual os $E$), we take the following procedure, different from the usual abstract approach (as given - for instance - in the proof of th IV - 11 - [2]), by making the hypothesis that the restriction of the Schwinger Generating Functional eq.(1) to any finite-dimensional $R^N$ of $E$ is the Fourier Transform of a positive continuous function, namely.

$$Z\left( \sum_{\alpha=1}^{N} C_\alpha \bar{j}_\alpha(x) \right) = \int_{R^N} \exp\left( i \sum_{\alpha=1}^{N} C_\alpha P_\alpha \right) \tilde{g}(P_1, \ldots, P_N) dP_1, \ldots, dP_N \tag{2}$$

Here $\{\vec{j}_\alpha(x)\}_{\alpha=1,\cdots,N}$ is a fixed vectorial base of the given finite-dimensional sub-space (isomorphic to $R^N$) of $E$.

As a consequence of the above made hypothesis (based physically on the Renormalizability and Unitary of the associated Quantum Field Theory), one can apply the Bochner - Martin - Kolmogorov Theorem ([5] - theorem 4.35) to write eq.(1) as a Functional Fourier

Transform on the Space $E^{alg}$ (see appendix A)

$$Z[j(x)] = \int_{E^{alg}} \exp(ih(j(x))d\mu(h) \tag{3}$$

where $d\mu(h)$ is the Kolmogorov cylindrical measure on $E^{alg} = \Pi_{\lambda \in A}(R^{\lambda})$ with $A$ denoting the index set of the fixed Hamel Vectorial Basis used in eq.(2) and $h(j(x))$ is the action of the given Linear (algebric) Functional (belonging to $E^{alg}$) on the element $j(x) \in E$.

At this point, we relate the mathematically non-rigorous physicist point of view to the Kolmogorov measure $d\mu(h)$ eq.(3) over the Algebraic Linear Functions on the Schwinger Source Space. It is formally given by the famous Feynman formulae when one identifies the action of $h$ on $E$ by means of an "integral" average

$$h(j) = \int_{R^D} dx^D j(x)h(x) \tag{4}$$

Formally we have the equation

$$d\mu(h) = \left( \prod_{x \in R^D} dh(x) \right) \exp\{-S(h(x))\} \tag{5}$$

where $S$ is the classical action of the Classical Field Theory under quantization, but with the necessary coupling constant renormalizations need to make the associated Quantum Field Theory well-defined.

Let us outline these proposed steps on a $\lambda\phi^4$ - Field Theory on $R^4$.

At first we will introduce the massive free field theory generating functional directly in the infinite volume space $R^4$.

$$Z[j(x)] = \exp\left\{ -\frac{1}{2} \int d^4x d^4x' j(x)((-\Delta)^{\alpha} + m^2)^{-1}(x, x')j(x') \right\} \tag{6}$$

where the Free Field Propagator is given by

$$((-\Delta)^{\alpha} + m^2)^{-1}(x, x') = \int d^4k \frac{e^{ik(x-x')}}{k^{2\alpha} + m^2} \tag{7}$$

with $\alpha$ a regularizing parameter with $\alpha > 1$.

As the source space, we will consider the vector space of all real sequences on $\Pi_{\lambda\in(-\infty,\infty)}(R)^{\lambda}$, but with only a finite number of non-zero components. Let us define the following family of finite-dimensional Positive Linear Functionals $\{L_{\Lambda_f}\}$ on the Functional Space $C(\prod_{\lambda\in(-\infty,\infty)} R^{\lambda}; R)$

$$L_{\Lambda_f}\left(e(P_{\lambda_{s_1}},\cdots P_{\lambda_{s_N}})\right)=\int_{(\prod_{\lambda\in\Lambda_f} R^{\lambda})} g(P_{\lambda_{s_1}},\ldots,P_{\lambda_{s_N}})\exp\left\{-\frac{1}{2}\sum_{\lambda\in\Lambda_f}(\lambda^{2\alpha}+m^2)(P_\lambda)^2\right\}$$
$$\left(\prod_{\lambda\in\Lambda_f} d(P_\lambda\sqrt{\pi(\lambda^{2\alpha}+m^2)})\right) \tag{8}$$

Here $\Lambda_f=\{\lambda_{s_1},\cdots,\lambda_{s_N}\}$ is an ordered sequence of real number of the real line which is the index set of the Hamel Basis of the Algebraic Dual of the proposed source space.

Note that we have the generalized eingenproblem expansion

$$((-\Delta)^{\alpha}+m^2)e^{i\lambda x}=(\lambda^{2\alpha}+m^2)e^{i\lambda x} \tag{9}$$

By the Stone-Weirstrass Theorem or the Kolmogoroff Theorem applied to the family of finite dimensional measure in eq.(8), there is a unique extension measure $d\mu(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$ to the space $\Pi_{\lambda\in(-\infty,\infty)}R^{\lambda}=E^{alg}$ and representing the Infinite-volume Generating Functional on our chosen source space (the usual Riesz-Markov theorem applied to the linear functional $L=\lim_{\{\Lambda_f\}}\sup L_{\Lambda_f}$, on $C(\Pi_{\lambda\in(-\infty,\infty)}R^{\lambda},R)$ leads to this extension measure ) ([10]).

$$Z[j(x)]=Z[\{j_\lambda\}_{\lambda\in\Lambda_f}]=\int_{\prod_{\lambda\in(-\infty,\infty)} R^{\lambda}} d\mu^{(0)\cdot(\alpha)}(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$$
$$\times\exp\left(i\sum_{\lambda\in(-\infty,\infty)} j_\lambda P_\lambda\right)=\exp\left\{-\frac{1}{2}\sum_{\lambda\in\Lambda_f}\frac{(j_\lambda)^2}{\lambda^{2\alpha}+m^2}\right\} \tag{10}$$

At this point it is very important remark that the generating functional eq.(10) has continuous natural extension to any test space ( $S(R^N),D(R^N)$, etc) which contains the continuous functions of compact support as a dense sub-space.

At this point we consider the following Quantum Field interaction functional which is a measurable functional in relation to the above constructed Kolmogoroff measure $d\mu^{(0)\cdot(\alpha)}(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$ for $\alpha$ non integer in the original field variable $\phi(x)$

$$V^{(\alpha)}(\phi) = \lambda_R\phi^4 + \frac{1}{2}(Z_\phi^{(\alpha)}(\lambda_R, M) - 1)\phi((-\Delta)^\alpha)\phi - \frac{1}{2}[(m^2 Z_\phi^{(\alpha)}(\lambda_R, m) - 1 \\ - (\delta m^2)^{(\alpha)}(\lambda_R)]\phi^2 - [Z_\phi^{(\alpha)}(\lambda_R, m)(\delta^{(\alpha)}\lambda)(\lambda_R, m)]\phi^4 \tag{11}$$

Here the renormalization constants are given in the usual analytical finite-part regularization form for a $\lambda\phi^4$ - Field Theory. It still a open problem in the mathematical-physics of quantum fields to prove the integrability in some Distributional space of the cut-off removing $\alpha \to 1$ limit of the interaction lagrangean $\exp(-V^{(\alpha)}(\phi))$ (see section 4 for a analysis of this cut off removing on space of functions).

# 3 The Support of Functional Measures - The Minlos Theorem.

Let us now analyze the measure support of Quantum Field Theories generating functional eq.(3).

For higher dimensional space-time, the only available result in this direction is the case that we have a Hilbert structure on $E$([4], [5], [6]).

At this point of our paper, we introduce some definitions. Let $\varphi : \mathbb{Z}^+ \to R$ be an increasing fixed function (including the case $\varphi(\infty) = \infty$). Let $E$ be denoted by $H$ and $H^Z$ be the sub-space of $H^{alg} = (\Pi_{\lambda\in A=[0,1]} R^\lambda)$ (with $A$ being the index set of a Hamel basis of $H$), formed by all sequences $\{x_\lambda\}_{\lambda\in A} \in H^{alg}$ with coordinates different from zero at most a countable number

$$H^Z = \{(x_\lambda)_{\lambda\in A|x_\lambda} \neq 0 \text{ for } \lambda \in \{\lambda_\mu\}_{\mu\in\mathbb{Z}}\} \tag{12}$$

Consider the following weighted sub-set of $H^{alg}$

$$H^Z_{(e)} = \{\{x_\lambda\}_{\lambda\in A} \in H^Z|$$

and

$$\lim_{N\to\infty}\left\{\frac{1}{\varphi(N)}\sum_{n=1}^{N}(x_{\lambda\sigma(\mu)})^2\right\}<\infty\}$$

for any $\sigma : N \to N$, a permutation of the natural numbers.

We now state our generalization of the Minlos Theorem.

**Theorem 3.** Let $T$ be an operator, with Domain $D(T) \subset H$, and $T; D(T) \to H$ such that for any finite-dimensional space $H^N \subset H$, the sum is bounded by the function $\varphi(N)$

$$\left(\sum_{(i,j)=1}^{N}\langle Te_i, Te_j\rangle^{(0)}\right) \le \varphi(N) \tag{13}$$

Here $\langle,\rangle^{(0)}$ is the inner product of $H$ and $\{e_p\}_{1\le p\le N}$ is a vectorial basis of the sub-space $H^N$ with dimension $N$.

Suppose that $Z[j(x)]$ is a continuous function an $D(T) = \overline{(D(T), \langle,\rangle^{(1)}})$ where $\langle,\rangle^{(1)}$ is a new inner product defined by the operator $T(\langle j, \overline{j}\rangle^{(1)} = \langle Tj, T\overline{j}\rangle^{(0)})$ we have, thus, that the support of the cylindrical measure eq.(3) is the measurable set $H_e^Z$.

**Proof:** following closely references ([1]) - Theorem 2.2., [4]) let us consider the following representation for the characteristic function of the measurable set $H_e^Z \subset H^{alg}$

$$\begin{aligned}
&\cdot X_{H_e^Z}(\{x_\lambda\}_{\lambda\in A}) = \\
&\lim_{\alpha\to 0}\lim_{N\to\infty}\exp\left\{-\frac{1\alpha}{2\varphi(N)}\sum_{\ell=1}^{N}x_{\lambda_l}^2\right\} \\
&= 1 \qquad \text{if} \qquad \lim_{N\to\infty}\frac{1}{\varphi(N)}\sum_{\ell=1}^{N}x_{\lambda_l}^2<\infty \\
&0 \qquad \text{otherwise}
\end{aligned} \tag{14}$$

Now its measure satisfies the following inequality

$$\int_{H^{alg}} d\mu(h) = \mu(H^{alg}) = 1 > \mu(H_e^Z) \tag{15}$$

But

$$\mu(H_e^Z) = \lim_{\alpha\to 0}\lim_{N\to\infty}\int\limits_{H^{alg}} d\mu(h)\exp - \left\{\frac{\alpha}{2\varphi(N)}\sum_{\ell=1}^{N} x_{\lambda_\ell}^2\right\} =$$

$$\lim_{\alpha\to 0}\lim_{N\to\infty}\left\{\frac{1}{(\frac{2\pi\alpha}{\varphi(N)})^{N/2}}\right\}\int_{R^N} dj_1,\cdots dj_N \tag{16}$$

$$\exp\left(-\frac{1}{2}\left(\frac{\varphi(N)}{\alpha}\right)\sum_{\ell=1}^{N} j_\ell^2\right)\tilde{Z}(j_1,\cdots,J_N)$$

where

$$\tilde{Z}(j_1,\cdots,j_N) = \int_{\substack{\pi R^\lambda\\ \lambda\in A}} d\mu(\{x_\lambda\})\exp\left(i\sum_{\ell=1}^{N} x_{\lambda_\ell} j_\ell\right) = \int\limits_{H^{alg}} d\mu(h)\exp\left(i\sum_{\ell=1}^{N} x_{\lambda_\ell} j_\ell\right) \tag{17}$$

Now due to the continuity and positivity of $Z[j]$ in $D(T)$; we have that for any $\varepsilon > 0 \to \exists\delta$ such that the inequality below is true since we have that: $Z(j_1,\cdots j_N) \geq 1-\varepsilon - \frac{2}{\delta^2}(j,j)^{(1)}$

$$\frac{1}{\left(\frac{2\pi\alpha}{\varphi(N)}\right)^{N/2}}\int\limits_{R^N} dj_1\cdots dj_N \exp\left(-\frac{1}{2}\frac{\varphi(N)}{\alpha}\sum_{\ell=1}^{N} j_\ell^2\right)\tilde{Z}(j_1\cdots,j_N)$$

$$\geq 1-\varepsilon-\frac{2}{\delta^2}\left\{\sum_{(m,n)=1}^{N}\frac{1}{\left(\frac{2\pi\alpha}{\varphi(N)}\right)^{N/2}}\int\limits_{R^N} dj_1\cdots dj_N\exp\left(-\frac{1}{2}\left(\frac{\varphi(N)}{\alpha}\right)\sum_{\ell=1}^{N} j_\ell^2\right) j_m j_n < e_m, e_n >^{(1)}\right\}$$

(18)

$$= 1-\varepsilon-\frac{2}{\delta^2}\left\{\left(\frac{\alpha}{\varphi(N)}\right)\sum_{(m,n)=1}^{N}\delta_{mn} < Te_n, Te_m >^{(0)}\right\}$$

$$\geq 1-\varepsilon-\frac{2}{\delta^2}\left(\frac{\alpha}{\varphi(N)}\right)\varphi(N) \geq 1-\varepsilon-\frac{2}{\delta^2}\alpha$$

By substituting eq.(18) into eq.(15), we get the result

$$1 \geq \mu(H_e^Z) \geq 1-\varepsilon-\frac{2}{\delta^2}\left(\lim_{\alpha\to 0}\alpha\right) = 1-\varepsilon \tag{19}$$

Since $\varepsilon$ was arbitrary we have the validity of our theorem.

As a consequence of this Theorem in the case of $\varphi(N)$ being bounded (so $TT^*$ is an operator of Trace Class), we have that $H_e^Z = H$ which is the usual Topological Dual of $H$.

At this point, a simple proof may be given to the usual Minlos Theorem on Schwartz Spaces ([5], [6],).

Let us consider $S(R^D)$ represented as the countable normed spaces of sequences ([8])

$$S(R^D) = \bigcap_{m=0}^{\infty} \ell_m^2 \tag{20}$$

where

$$\ell_m^2 = \{(x_n)_{n\in\mathbb{Z}}, x_n \in R \Big| \sum_{n=0}^{N} (x_n)^2 n^m < \infty\} \tag{21}$$

The Topological Dual is given by the nuclear structure sum ([8])

$$S'(R^D) = \bigcup_{n=0}^{\infty} \ell_{-n}^2 = \bigcup_{n=0}^{\infty} (\ell_n^2)^* \tag{22}$$

We, thus, consider $E = S(R^D)$ in eq.(3) and $Z[j(x)] = Z[\{j_n\}_{n\in\mathbb{Z}}]$ as a continuous on $\bigcap_n^{\infty} \ell_n$. Since $Z[\{j_n\}_{n\in\mathbb{Z}}] \in C(\bigcap_{n=0}^{\infty} \ell_n^2, R)$ we have that for any fixed integer $p, Z[\{j_n\}_{n\in\mathbb{Z}}]$ is continuous on the Hilbert Space $\ell_p^2$ which, by its turn, may be considered as the Domain of the following operator.

$$\begin{aligned} T_p : \ell_p^2 c \ell_0 &\to \ell_0 \\ \{j_n\} &\to \{n^{p/2} j_n\} \end{aligned} \tag{23}$$

It is straightforward to have the estimate

$$\Big| \sum_{(m,n)=1}^{N} \langle T_p e_m \, , \, T_p e_n \rangle^{(0)} \Big| \leq N^{(B_p)} \tag{24}$$

for some positive integer $B$ and $\{e_i\}$ being the canonical orthonormal basis of $l_o^2$. By an application of our theorem for each fixed $p$; we get that the support of measure is given by the union of weighted spaces

$$\text{supp} d\mu(h) = \bigcup_{p=0}^{\infty} (\ell_p^2)^* = \bigcup_{p=0}^{\infty} \ell_{-p}^2 = S'(R^D) \tag{25}$$

At this point we can suggest, without a proof a straightforward (non topological) generalization of the Minlos Theorem.

**Theorem 4.** Let $\{T_\beta\}_{\beta\in C}$ be a family of operators satisfying the hypothesis of Theorem 3. Let us consider the Locally Convex space $\bigcup\limits_{\beta\in C}\overline{Dom(T_\beta)}$ (supposed non-empty) with the family of norms $\| \psi \|_\beta = < T_\beta\psi, T_\beta\psi >^{1/2}$

If the Functional Fourier Transform is continuous on this Locally Convex Space, the support of the Kolmogoroff measure eq.(3) is given by the following sub-set of $[\bigcup\limits_{\beta\in C}\overline{Dom(T_\beta)]^{alg}}$, namely

$$\operatorname{supp} d\mu(h) = \bigcup_{\beta\in C} H^2_{\varphi_\beta} \tag{26}$$

where $\varphi_\beta$ are the functions given by Theorem 3. This general theorem will not be applied in what follows.

Let us now proceed to apply the above displayed results by considering the Schwinger Generating Functionals for two-dimensional Euclidean Quantum Eletrodynamics in Bosonized Parametrization ([9])

$$Z[j(x)] = \exp\left\{-\frac{1}{2}\int_{R^2} d^2x \int_{R^2} d^2y j(x)((-\Delta)^2 + \frac{e^2}{\pi}(-\Delta))^{-1}(x,y)j(y)\right\} \tag{27}$$

where in eq.(27), the electromagnetic field has the decomposition in Landau Gauge

$$A_\mu(x) = (\varepsilon_{\mu\nu}\partial_\nu\phi)(x) \tag{28}$$

and $j(x)$ is, thus, the Schwinger Source for the $\phi(x)$ field taken as a basic dynamical variable ([9]).

Since eq.(27) is continuous in $L^2(R^2)$ with the inner product defined by the trace class operator $((-\Delta)^2 + \frac{e^2}{\pi}(-\Delta))^{-1}$, we conclude on basis of theorem 3 that the associated Kolmogoroff measure in eq.(3) has its support in $L^2(R^2)$ with the usual inner product. As a consequence, the Quantum Observable Algebra will be given by the Functional Space $L^1(L^2(R^2), d\mu(h))$ and usual orthonormal Finite - Dimensional approximations in Hilbert

Spaces may be used safely i.e if one considers the basis expansion $h(x) = \sum_{n=1}^{\infty} h_n e_n(x)$ with $e_n(x)$ denoting the eigenfunctions of the operator in eq.(27) we get the result

$$\bigcup_{n=1}^{\infty} L^1(R^N, d\mu(h_1, \cdots, h_N)) = L^1(L^2(R^2), d\mu(h)) \tag{29}$$

It is worth mentioning that if one uses the Gauge Vectorial Field parametrization for the $(Q.E.D)_2$ - Schwinger Functional

$$Z[j_1(x), j_2(x)] = \exp\left\{ -\frac{1}{2} \int_{R^2} d^2x \int_{R^2} d^2y j_i(x) \left( -\Delta + \frac{e^2}{\pi} \right)^{-1} (x,y) \delta_{il} j_l(x) \right\} \tag{30}$$

the associated measure support will now be the Schwartz Space $S'(R^2)$ since the operator $(-\Delta + \frac{e^2}{\pi})^{-1}$ is an application of $S(R^2)$ to $S'(R^2)$. As a consequence it will be very cumbersome to use Hilbert Finite Dimensional approximations ([8]) as in eq.(29).

An alternative to approximate tempered distributions is the use of its Hermite expansion in $S'(R)$ distributional space associated to the eigenfunctions of the Harmonic-oscillator $V(x) \in L^{\infty}(R) \bigcup L^2(R)$ potential pertubation (see ref. [3] for details with $V(x) \equiv 0$).

$$(-\frac{d^2}{dx^2} + x^2 + V(x)) H_n(x) = \lambda_n H_n(x) \tag{31}$$

Another important class of Bosonic Functionals Integrals are those associated with an Elliptic Positive Self-Adjoint Operator $A^{-1}$ on $L^2(\Omega)$ with suitable Boundary conditions. Here $\Omega$ denotes a $D$-dimensional compact manifold of $R^D$ with volume element $d\nu(x)$.

$$Z[j(x)] = \exp\left\{ -\frac{1}{2} \int_{\Omega} d\nu(x) \int_{\Omega} d\nu(y) j(x) A^{+1}(x,y) j(y) \right\} \tag{32}$$

If $A$ is an operator of trace class on $(L^2(\Omega), d\nu)$ we have, thus, the validity of the usual eigenvalue Functional Representation

$$Z[\{j_n\}_{\mu\in\mathbb{Z}}] = \int \left( \prod_{\ell=1}^{\infty} d(c_\ell \sqrt{\lambda_\ell}) \right) \exp\left( -\frac{1}{2} \sum_{\ell=1}^{\infty} \lambda_\ell c_\ell^2 \right) X_{\ell^2}(\{c_n\}_{n\in\mathbb{Z}}) \exp(i \sum_{\ell=1}^{\infty} c_\ell j_\ell)$$

with the spectral set

$$A^{-1}\sigma_\ell = \lambda_\ell \sigma_\ell$$
$$j_\ell = \langle j, \sigma_\ell \rangle \tag{33}$$

and the characteristic function set

$$X_{\ell^2}(\{c_n\}_{n\in\mathbb{Z}}) = \begin{cases} 1 \text{ if} & \sum\limits_{n=0}^{\infty} c_n^2 < \infty \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

It is instructive point out the usual Hermite functional basis (see 5.4 - [5]) are a complete set in $L^2(E^{alg}, d\mu(h))$, only if the Gaussian Kolmogoroff measure $d\mu(h)$ is of the class above studied

A criticism to the usual framework to construct Euclidean. Field Theories is that is very cumbersome to analyze the infinite volume limit from the Schwinger Generating Functional defined originally on Compact Space Times. In two dimensions the use of the result that the massive Scalar Field Theory Generating Functional

$$\exp\left\{-\frac{1}{2}\int\limits_{R^2} d^2x \int\limits_{R^2} d^2y j(x)(-\Delta + m^2)^{-1}(x,y)j(y)\right\} \tag{35}$$

with $j(x) \in S(R^2)$; is given by the limit of Finite Volume Dirichlet Field Theories

$$\lim_{\substack{L\to\infty \\ T\to\infty}} \exp\Big\{ -\frac{1}{2}\int_{-L}^{L} dx^0 \int_{-T}^{T} dx^1 \int_{-L}^{L} dy^0$$
$$\int_{-T}^{T} dy^1 j(x^0,x^1)(-\Delta_D + m^2)^{-1}(x^1,y^1,x^0,y^0)j(y^0,y^1) \tag{36}$$

may be considered, in our opinion, as the similar claim made that is possible from a mathematical point of view to deduce the Fourier Transforms from Fourier Series, a very, difficult mathematical task (see appendix B).

Let us comment on the functional integral associated to Feynman propagation of fields

configurations used in geometrodynamical theories in the scalar case

$$G[\beta^{in}(x);\beta^{out}(x),T](j) = \int_{\substack{\phi(x,0)=\beta^{in}(x)\\ \phi(x,T)=\beta^{out}(x)}}$$
$$\exp\left\{-\frac{1}{2}\int_0^T dt \int_{-\infty}^{+\infty} d^\nu x \left(\phi\left(-\frac{d^2}{dt^2}+A\right)\phi\right)(x,t)\right\}$$
$$\exp\left(i\int_0^T dt \int_{-\infty}^{+\infty} d^\nu x j(x)t)\phi(x,t)\right) \tag{37}$$

If we define the formal functional integral by means of the eigenfunctions of the self-adjoint Eliptic operator $A$, namely:

$$\phi(x,t) = \sum_{\{k\}} \phi_k(t)\psi_k(x) \tag{38}$$

where

$$A\psi_k(x) = (\lambda_k)^2\psi_k(x) \tag{39}$$

it is straightforward to see that eq.(36) is formally exactly evaluated in terms of an infinite product of usual Feynman Wiener - path measures

$$G[\beta^{in}(x);\beta^{out}(x),T](j) =$$
$$= \prod_{\{k\}} \int_{\substack{c_k(0)=\phi_k(0)\\ c_k(T)=\phi_k(T)}} D^F[c_k(t)] \exp\left\{-\frac{1}{2}\int_0^T \left(c_k\left(-\frac{d^2}{dt^2}+\lambda_k^2\right)c_k\right)(t)dt \exp\left(i\int_0^T dtj_k(t)c_k(t)\right)\right\}$$
$$= \prod_{\{k\}} \left\{\sqrt{+\frac{\lambda_k}{\sin(\lambda_k T)}} \exp\left\{-\frac{\lambda_k}{2\sin(\lambda_k T)}\Big[(\phi_k^2(T)+\phi_k^2(0))\cos(\lambda_k T) - 2\phi_k(0)\phi_k(T)\right\}\right\}$$
$$-\frac{2\phi_k(T)}{\lambda_k}\int_0^T dtj_k(t)\sin(\lambda_k t) - \frac{2\phi_k(0)}{\lambda_k}\int_0^T dtj_k(t)\sin(\lambda_k(T-t))$$
$$\left\{-\frac{2}{(\lambda_k)^2}\int_0^T dt\int_0^t dsj_k(t)j_k(s)\sin(\lambda_k(T-t))\sin(\lambda_k s)\Big]\right\} \tag{40}$$

Unfortunately, our theorems do not aply in a straightforward way to infinite (continuum) measure product of Wiener measures in eq.(40) to produce a sensible measure theory on the functional space of the infinite product of Wiener trajectories $\{c_k(t)\}$ (Note

that for each $x$ fixed, a sample field configuration $\phi(t,0)$ in eq.(36) is a Hölder continuous function, result opposite to the usual functional integral representation for the Schwinger generating functional eq.(1)- eq.(5)) where it does not make a mathematical sense to consider a fixed point distribution $\phi(t,0)$ - see section 4 - eq.(74).

Let us call attention that still there is a formal definition of the above Feynman Path propagator for fields eq.(37) which at large time $T \to +\infty$ gives formally the Quantum Field Functional integral eq.(5) associated to the Schwinger Generating Functional.

We thus consider the functional domain for eq.(37) as composed of field configurations which has a classical piece added with another fluctuating component to be functionally integrated out, namely

$$\sigma(x,t) = \sigma_{CL}(x,t) + \sigma_q(x,t) \tag{41}$$

Here the classical field configuration problem (added with all zero modes of the free theory) defined by the kinetic term $\pounds$

$$\left(-\frac{d^2}{dt^2} + \pounds\right)\sigma^{CL}(x,t) = j(x,t) \tag{42}$$

with

$$\sigma^{CL}(x,-T) = \beta_1(x); \sigma^{CL}(x,T) = \beta_2(x) \tag{43}$$

namely

$$\sigma_{CL}(x,t) = \left(-\frac{d^2}{dt^2} + \pounds\right)^{-1} j(x,t) + (\text{ all projection on zero modes of } \pounds) \tag{44}$$

As a consequence of the decomposition eq.(41), the formal geometrical propagator with an external source below

$$\begin{aligned} & G[\beta_1(x),\beta_2(x),T,[j]] \\ & = \int_{\substack{\sigma(x,-T)=\beta_1(x) \\ \sigma(x,+T)=\beta_2(x)}} D[\sigma(x,t)] \exp\left(-\frac{1}{2}\int_{-T}^{T} dt d^\nu x \sigma(x,t)\left(-\frac{d^2}{dt^2}+\pounds\right)\sigma(x,t)\right) \\ & \exp(i\int_{-T}^{T} dt \int d^\nu x j(x,t)\sigma(x,t)) \end{aligned} \tag{45}$$

may be defined the following mathematically well defined Gaussian functional measure

$$\exp\left\{-\frac{1}{2}\int_{-T}^{T}dt\int d^{\nu}xj(x,t)\sigma^{CL}(x,t)\right\}\times$$
$$\int_{\substack{\sigma_q(x,-T)=0\\ \sigma_q(x,+T)=0}} d\sigma_q(x,t)\exp\left\{-\frac{1}{2}\int_{-T}^{T}dt\int d^{\nu}x\sigma_q(x,t)\left(-\frac{d^2}{dt^2}+\pounds\right)\sigma_q(x,t)\right\} \tag{46}$$

The above claim is a consequence of the result below

$$\int_{\substack{\sigma_q(x,-T)=0\\ \sigma_q(x,T)=0}} D[\sigma_q(x,t)]\exp\left\{-\frac{1}{2}\int_{-T}^{T}dt\int d^{D}x\sigma_q(x,t)\left(-\frac{d^2}{dt^2}+\pounds\right)\sigma_q(x,t)\right\}$$
$$= det_{Dir}^{-\frac{1}{2}}\left[-\frac{d^2}{dt^2}+\pounds\right] \tag{47}$$

where the sub-script Dirichlet on the functional determinant means that one must impose formally the Dirichlet condition on the domain of the operator $\left(-\frac{d^2}{dt^2}+\pounds\right)$ on $D'(R^D\times[-T,T])$ (or $L^2(R^D\times[-T,T]$ if $\pounds^{-1}$ belongs to trace class). Note that the operator $\pounds$ in eq.(46) does not have zero modes by the construction of eq.(41).

At this point, we remark that at the limit $T\to+\infty$ eq.(45) is exactly the Quantum Field functional eq.(5) if one takes $\beta_1(x)=\beta_2(x)=0$ (Note that the classical vacuum limit $T\to\infty$ of Wiener measures is mathematically ill-defined (see theorem 5.1. of ref [1]).

It is a important point to remark that $\sigma_{CL}(x,t)$ is a regular $C^\infty([-T,T]\times\Omega)$ solution of the Eliptic problem eq.(42) and the fluctuating component $\sigma_q(x,t)$ is a Schwartz distribution in view of the Minlos - Dao Xing theorem 3, since the Eliptic operator $-\frac{d^2}{dt^2}+\pounds$ in eq.(47) acts now on $D'([-T,T]\times\Omega)$ with range $D([-T,T]\times\Omega)$, which by its turn shows the difference between this framework and the previous one related to the infinite product of Wiener measures since these objects are functional measures in different Functional Spaces

Finally we comment that Functional Schrodinger equation, may be mathematically defined for the above displayed field propagators eq.(37) only in the situation of eq.(40). For instance, with $\pounds=-\Delta$ (the Laplacean), we have the validity of the Euclidean field

wave equation for the Geometrodynamical path-integral eq (37)

$$\begin{aligned}&\frac{\partial}{\partial T}G[\beta_1(x),\beta_2(x),T,[j]] = \\ &= \int_\Omega d^\nu x\Big[+\frac{\delta^2}{\delta^2\beta_2(x)} - |\nabla\beta_2(x)|^2 + j(x,T)\Big]G[\beta_1(x),\beta_2(x),T,[j]]\end{aligned} \tag{48}$$

with the functional initial - condition

$$\lim_{T\to 0^+} G[\beta_1(x),\beta_2(x),T] = \delta^{(F)}(\beta_1(x)-\beta_2(x)) \tag{49}$$

# 4 Some rigorous quantum field path integral in the Analytical regularization scheme

In this core section of our paper we address the important problem of producing concrete non-trivial examples of mathematically well - defined (in the ultra - violet region!) path integrals in the context of the exposed theorems on the previously sections of this paper, specially section 2 - eq.(11).

Let us thus start our analysis by considering the Gaussian measure associated to the (infrared regularized) $\alpha$-power ($\alpha > 1$) of the Laplacean acting on $L^2(R^2)$ as an operational quadratic form (the Stone spectral theorem)

$$(-\Delta)^\alpha_\varepsilon = \int_{\varepsilon_{IR}\le\lambda} (\lambda)^\alpha \, dE(\lambda) \tag{50-a}$$

$$\begin{aligned}Z^{(0)}_{\alpha,\varepsilon_{IR}}[j] &= \exp\left\{-\frac{1}{2}\Big\langle j, (-\Delta)^{-\alpha}_\varepsilon \, j\Big\rangle_{L^2(R^2)}\right\} \\ &= \int d^{(0)}_{\alpha,\varepsilon}\mu\,[\varphi]\,\exp\left(i\Big\langle j,\varphi\Big\rangle_{L^2(R^2)}\right)\end{aligned} \tag{50-b}$$

Here $\varepsilon_{IR} > 0$ denotes the infrared cut off.

It is worth call the reader attention that due to the infrared regularization introduced on eq (50-a), the domain of the Gaussian measure is given by the space of square integrable

functions on $R^2$ by the Minlos theorem of section 3, since for $\alpha > 1$, the operator $(-\Delta)^{-\alpha}_{\varepsilon_{IR}}$ defines a classe trace operator on $L^2(R^2)$, namely

$$Tr_{\oint_1}((-\Delta)^{-\alpha}_{\varepsilon_{IR}}) = \int d^2k \frac{1}{(|K|^{2\alpha} + \varepsilon_{IR})} < \infty \tag{50-c}$$

This is the only point of our analysis where it is needed to consider the infra-red cut off considered on the spectral resolution eq (50-a). As a consequence of the above remarks, one can analize the ultra-violet renormalization program in the following interacting model proposed by us and defined by an interaction $g_{\text{bare}}V(\varphi(x))$, with $V(x)$ denoting a compact support function on $R$ such, that it posseses an essentially bounded Fourier transform and $g_{\text{bare}}$ denoting the positive bare coupling constant.

Let us show that by defining a renormalized coupling constant as (with $g_{ren} < 1$)

$$g_{\text{bare}} = \frac{g_{\text{ren}}}{(1-\alpha)^{1/2}} \tag{51}$$

one can show that the interaction function

$$\exp\left\{-g_{\text{bare}}(\alpha)\int d^2x\, V(\varphi(x))\right\} \tag{52}$$

is an integrable function on $L^1(L^2(R^2), d^{(0)}_{\alpha,\varepsilon_{IR}}\mu\,[\varphi])$ and leads to a well-defined ultra-violet path integral in the limit of $\alpha \to 1$.

The proof is based on the following estimates.

Since almost everywhere we have the pointwise limit

$$\exp\left\{-g_{\text{bare}}(\alpha)\int d^2x\, V(\varphi(x))\right\}$$

$$\lim_{N\to\infty}\left\{\sum_{n=0}^{N}\frac{(-1)^n(g_{\text{bare}}(\alpha))^n}{n!}\int_R dk_1\cdots dk_n\,\tilde V(k_1)\cdots\tilde V(k_n)\int_{R^2}dx_1\cdots dx_n\,e^{ik_1\varphi(x_1)}\cdots e^{ik_n\varphi(x_n)}\right\} \tag{53}$$

we have that the upper-bound estimate below holds true

$$\begin{aligned}\left|Z^{\alpha}_{\varepsilon_{IR}}[g_{\text{bare}}]\right| \le \Big|\sum_{n=0}^{\infty}\frac{(-1)^n(g_{\text{bare}}(\alpha))^n}{n!}&\int_R dk_1\cdots dk_n\,\tilde V(k_1)\cdots\tilde V(k_n)\\ &\int_{R^2}dx_1\cdots dx_n\int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi](e^{i\sum_{\ell=1}^{N}k_\ell\varphi(x_\ell)})\Big|\end{aligned} \tag{54-a}$$

with

$$Z^{\alpha}_{\varepsilon_{IR}}[g_{\text{bare}}] = \int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi]\,\exp\left\{-g_{\text{bare}}(\alpha)\int d^2x\,V(\varphi(x))\right\} \tag{54-b}$$

we have, thus, the more suitable form after realizing the $d^2k_i$ and $d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi]$ integrals respectivelly

$$\begin{aligned}\left|Z^{\alpha}_{\varepsilon_{IR}=0}[g_{bare}]\right| \le \sum_{n=0}^{\infty}\frac{(g_{bare}(\alpha))^n}{n!}\left(||\tilde{V}||_{L^{\infty}(R)}\right)^n \\ \left|\int dx_1\cdots dx_n\,\det{}^{-\frac{1}{2}}\left[G^{(N)}_{\alpha}(x_i,x_j)\right]_{\substack{1\le i\le N\\1\le j\le N}}\right|\end{aligned} \tag{55}$$

Here $[G^{(N)}_{\alpha}(x_i,x_j)]_{\substack{1\le i\le N\\1\le j\le N}}$ denotes the $N\times N$ symmetric matrix with the $(i,j)$ entry given by the Green-function of the $\alpha$-Laplacean (without the infra-red cut off here! and the needed normalization factors !).

$$G_{\alpha}(x_i,x_j) = |x_i-x_j|^{2(1-\alpha)}\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \tag{56}$$

At this point, we call the reader attention that we have the formulae on the asymptotic behavior for $\alpha\to 1$.

$$\left\{\lim_{\substack{\alpha\to1\\ \alpha>1}}\det{}^{-\frac{1}{2}}[G^{(N)}_{\alpha}(x_i,x_j)]\right\}\sim(1-\alpha)^{N/2}\times\left(\left|\frac{(N-1)(-1)^N}{\pi^{N/2}}\right|\right)^{-\frac{1}{2}} \tag{57}$$

After substituting eq.(57) into eq.(55) and taking into account the hypothesis of the compact support of the non-linearily $V(x)$ (for instance: $supp\,V(x)\subset[0,1]$), one obtains the finite bound for any value $g_{rem}>0$, and producing a proof for the convergence of the perturbative expansion in terms of the renormalized coupling constant.

$$\begin{aligned}\lim_{\alpha\to1}\left|Z^{\alpha}_{\varepsilon_{IR}=0}[g_{bare}(\alpha)]\right| &\le \sum_{n=0}^{\infty}\frac{(\|\tilde{V}\|_{L^{\infty}(R)})^n}{n!}\left(\frac{g_{ren}}{(1-\alpha)^{\frac{1}{2}}}\right)^n\times\frac{(1^n)}{\sqrt{n}}(1-\alpha)^{n/2} \\ &\le e^{g_{ren}\|\tilde{V}\|_{L^{\infty}(R)}}<\infty\end{aligned} \tag{58}$$

Another important rigorously defined functional integral is to consider the following $\alpha$-power Klein Gordon operator on Euclidean space-time

$$\mathcal{L} = (-\Delta)^{\alpha}+m^2 \tag{59}$$

with $m^2$ a positive "mass" parameters.

Let us note that $\mathcal{L}^{-1}$ is an operator of class trace on $L^2(R^\nu)$ if and only if the result below holds true

$$Tr_{L^2(R^\nu)}(\mathcal{L}^{-1}) = \int d^\nu k \frac{1}{k^{2\alpha}+m^2} = \bar{C}(\nu)\, m^{(\frac{\nu}{\alpha}-2)} \times \left\{ \frac{\pi}{2\alpha} \mathrm{cosec} \frac{\nu\pi}{2\alpha} \right\} < \infty \tag{60}$$

namely if

$$\alpha > \frac{\nu - 1}{2} \tag{61}$$

In this case, let us consider the double functional integral with functional domain $L^2(R^\nu)$

$$\begin{aligned} Z[j,k] = & \int d_G^{(0)}\, \beta[v(x)] \\ & \times \int d^{(0)}_{(-\Delta)^\alpha + v + m^2}\, \mu[\varphi] \\ & \times \exp\left\{ i \int d^\nu x\, (j(x)\, \varphi(x) + k(x)\, v(x)) \right\} \end{aligned} \tag{62}$$

where the Gaussian functional integral on the fields $V(x)$ has a Gaussian generating functional defined by a $\oint_1$-integral operator with a positive defined kernel $g(|x-y|)$, namely

$$\begin{aligned} Z^{(0)}[k] &= \int d_G^{(0)} \beta[v(x)] \exp\left\{ i \int d^\nu x\, k(x) v(x) \right\} \\ &= \exp\left\{ -\frac{1}{2} \int d^\nu x \int d^\nu y\, (k(x)\, g(|x-y|)\, k(x)) \right\} \end{aligned} \tag{63}$$

By a simple direct application of the Fubbini-Tonelli theorem on the exchange of the integration order on eq.(62), lead us to the effective $\lambda\varphi^4$ - like well-defined functional integral representation

$$\begin{aligned} Z_{\mathrm{eff}}[j] = & \int d^{(0)}_{((-\Delta)^\alpha + m^2)} \mu\, [\varphi(x)] \\ & \exp\left\{ -\frac{1}{2} \int d^\nu x d^\nu y\, |\varphi(x)|^2\, g(|x-y|)\, |\varphi(y)|^2 \right\} \\ & \times \exp\left\{ i \int d^\nu x\, j(x) \varphi(x) \right\} \end{aligned} \tag{64}$$

Note that if one introduces from the begining a bare mass parameters $m^2_{bare}$ depending on the parameters $\alpha$, but such that it always satisfies eq.(60) one should obtains again eq.(64) as a well-defined measure on $L^2(R^\nu)$. Of course that the usual pure Laplacean limit of $\alpha \to 1$ on eq.(59), will needed a renormalization of this mass parameters ($\lim\limits_{\alpha\to 1} m^2_{bare}(\alpha) = +\infty!$) as much as done in the previous example.

Let us continue our examples by showing again the usefulness of the precise determination of the functional - distributional structure of the domain of the functional integrals in order to construct rigorously these path integrals without complicated limit procedures.

Let us consider a general $R^\nu$ Gaussian measure defined by the Generating functional on $S(R^\nu)$ defined by the $\alpha$-power of the Laplacean operator $-\Delta$ acting on $S(R^\nu)$ with a of small infrared regularization mass parameter $\mu^2$

$$
\begin{aligned}
Z_{(0)}[j] &= \exp\left\{-\frac{1}{2}\left\langle j,\, ((-\Delta)^\alpha + \mu_0^2)^{-1}\, j\right\rangle_{L^2(R^\nu)}\right\} \\
&= \int_{E^{alg}(S(R^\nu))} d^{(0)}_\alpha\, \mu[\varphi]\, \exp(i\,\varphi(j))
\end{aligned}
\tag{65}
$$

An explicitly expression in momentum space for the Green function of the $\alpha$-power of $(-\Delta)^\alpha + \mu_0^2$ given by

$$
((-\Delta)^{+\alpha} + \mu_0^2)^{-1}(x-y) = \int \frac{d^\nu k}{(2\pi)^\nu}\, e^{ik(x-y)} \left(\frac{1}{k^{2\alpha} + \mu_0^2}\right) \tag{66}
$$

Here $\bar{C}(\nu)$ is a $\nu$-dependent (finite for $\nu$-values !) normalization factor.

Let us suppose that there is a range of $\alpha$-power values that can be choosen in such way that one satisfies the constraint below

$$
\int_{E^{alg}(S(R^\nu))} d^{(0)}_\alpha\, \mu[\varphi](\|\varphi\|_{L^{2j}(R^\nu)})^{2j} < \infty \tag{67}
$$

with $j = 1, 2, \cdots, N$ and for a given fixed integer $N$, the highest power of our polinomial field interaction. Or equivalently, after realizing the $\varphi$-Gaussian functional integration,

with a space-time cutt off volume $\Omega$ on the interaction to be analyzed on eq.(70)

$$\int_\Omega d^\nu x[(-\Delta)^\alpha + \mu_0^2]^{-j}(x,x) = \mathrm{vol}(\Omega) \times \left(\int \frac{d^\nu k}{k^{2\alpha}+\mu_0^2}\right)^j$$
$$= C_\nu(\mu_0)^{(\frac{\nu}{\alpha}-2)} \times \left(\frac{\pi}{2\alpha}\mathrm{cosec}\frac{\nu\pi}{2\alpha}\right) < \infty \tag{68}$$

For $\alpha > \frac{\nu-1}{2}$, one can see by the Minlos theorem that the measure support of the Gaussian measure eq.(65) will be given by the intersection Banach space of measurable Lebesgue functions on $R^\nu$ instead of the previous one $E^{alg}(S(R^\nu))$

$$\mathcal{L}_{2N}(R^\nu) = \bigcap_{j=1}^{N}(L^{2j}(R^\nu)) \tag{69}$$

In this case, one obtains that the finite - volume $p(\varphi)_2$ interactions

$$\exp\left\{-\sum_{j=1}^{N}\lambda_{2j}\int_\Omega (\varphi^2(x))^j\, dx\right\} \leq 1 \tag{70}$$

is mathematically well-defined as the usual pointwise product of measurable functions and for positive coupling constant values $\lambda_{2j} \geq 0$. As a consequence, we have a measurable functional on $L^1(\mathcal{L}_{2N}(R^\nu);\, d_\alpha^{(0)}\,\mu[\varphi])$ ( since it is bounded by the function 1). So, it would make sense to consider mathematically the well-defined path - integral on the full space $R^\nu$ with those values of the power $\alpha$ satisfying the contraint eq.(67).

$$Z[j] = \int_{\mathcal{L}_{2N}(R^\nu)} d_\alpha^{(0)}\,\mu[\varphi]\,\exp\left\{-\sum_{j=1}^{N}\lambda_{2j}\int_\Omega \varphi^{2j}(x)dx\right\} \times \exp(i\int_{R^\nu} j(x)\varphi(x)) \tag{71}$$

Finally, let us consider a interacting field theory in a compact space-time $\Omega \subset R^\nu$ defined by an iteger even power $2n$ of the Laplacean operator with Dirichlet Boundary conditions as the free Gaussian kinetic action, namely

$$Z^{(0)}[j] = \exp\left\{-\frac{1}{2}\Big\langle j, (-\Delta)^{-2n} j\Big\rangle_{L^2(\Omega)}\right\}$$
$$= \int_{W_2^n(\Omega)} d_{(2n)}^{(0)}\,\mu[\varphi]\,\exp(i\langle j,\varphi\rangle_{L^2(\Omega)}) \tag{72}$$

here $\varphi \in W_2^n(\Omega)$ - the Sobolev space of order $n$ which is the functional domain of the cylindrical Fourier Transform measure of the Generating functional $Z^{(0)}[j]$, a continuous bilinear positive form on $W_2^{-n}(\Omega)$ (the topological dual of $W_2^n(\Omega)$).

By a straightforward application of the well-known Sobolev immersion theorem, we have that for the case of

$$n - k > \frac{\nu}{2} \tag{73}$$

including $k$ a real number the functional Sobolev space $W_2^n(\Omega)$ is contained in the continuously fractional differentiable space of functions $C^k(\Omega)$. As a consequence, the domain of the Bosonic functional integral can be further reduced to $C^k(\Omega)$ in the situation of eq.(73)

$$Z^{(0)}[j] = \int_{C^k(\Omega)} d^{(0)}_{(2n)}\,\mu[\varphi]\ \exp(i\langle j,\varphi\rangle_{L^2(\Omega)}) \tag{74}$$

That is our new result generalizing the Wiener theorem on Brownian paths in the case of $n = 1$ , $k = \frac{1}{2}$ and $\nu = 1$

Since the bosonic functional domain on eq.(74) is formed by real functions and not distributions, we can see straightforwardly that any interaction of the form

$$\exp\left\{-g\int_\Omega F(\varphi(x))d^\nu x\right\} \tag{75}$$

with the non-linearity $F(x)$ denoting a lower bounded real function ($\gamma > 0$)

$$F(x) \geq -\gamma \tag{76}$$

is well-defined and is integrable function on the functional space $(C^k(\Omega),\, d^{(0)}_{(2n)}\,\mu[\varphi])$ by a direct application of the Lebesque theorem

$$\left|\exp\left\{-g\int_\Omega F(\varphi(x))\,d^\nu x\right\}\right| \leq \exp\{+g\gamma\} \tag{77}$$

At this point we make a subtle mathematical remark that the infinite volume limit of eq.(74) - eq.(75) is very difficult, since one looses the Garding - Poincaré inequalite at this limit for those elliptic operators and, thus, the very important Sobolev theorem.

The probable correct procedure to consider the thermodynamic limit in our Bosonic path integrals is to consider solely a volume cut off on the interaction term Gaussian action as in eq.(71) and there search for $\mathrm{vol}(\Omega) \to \infty$.

As a last remark related to eq.(73) one can see that a kind of "fishnet" exponential generating functional

$$Z^{(0)}[j] = \exp\left\{ -\frac{1}{2} \Big\langle j, \exp\{-\alpha\Delta\} j \Big\rangle_{L^2(\Omega)} \right\} \tag{78}$$

has a Fourier transformed functional integral representation defined on the space of the infinitelly differentiable functions $C^\infty(\Omega)$, which physically means that all field configurations making the domain of such path integral has a strong behavior like purely nice smooth classical field configurations.

As a general conclusion of this central section of our work, we can see that the technical knowledge of the support of measures on infinite dimensional spaces-specially the powerfull Minlos theorem of section 3 is very important for a deep mathematical physical understanding into one of the most important problem is Quantum Field theory and turbulence which is the problem related to the appearance of ultra-violet (short-distance) divergences on perturbative path integral calculations.

# 5 On the equilibrium measure for a non-linear diffusion equation: Some mathematical path integrals remarks

A very important conceptual issue on non-linear diffusion for Parabolic equations is the existence of an equilibrium probability distribution at large time when the whole system is subject to a contact with a reservoir at temperature $T$ ([11]). Let us comment on this difficult mathematical problem on this section. The stochastic evolution global equation governing such open system is given by the generalized Langevin equation for fields ([11])

with $0 < t < \infty$, $\Omega \subset R^\nu$ and $V(x)$ a Lipschitzion function - the problem's non-linearity.

$$\frac{\partial U(x,t)}{\partial t} = +(\Delta U)(x,t) - V(U)(x,t) + \eta(x,t) \tag{79}$$

$$U(x,0) = g(x) \in L^2(\Omega) \tag{80}$$

where $\eta(x,t)$ denotes a stirring (somewhat formal) white-noise stochastic process representing the effects of the thermal coupling between our non-linear diffusion field with an external thermal reservoir at temperature $T$. Its two-point function will be (generically) given by:

$$E_\eta[\eta(x,t)\,\eta(y,t')] = kT\,\delta(x-y)\,\delta(t-t') \tag{81}$$

Mathematically we can model such contact-reservoir noise as the (generalized - distributional sense) time derivative of an infinite statistically independent set of Brownian trajectories $\{b_i^1(t),\, 0 \le t \le T\}$ , namely ([12]).

$$\eta(x,t) = \sum_{l=1}^{\infty} \left(\frac{db_i}{dt}(t)\right) \varphi_i(x) \tag{82}$$

Let us re-wite the full non-linear stochastic diffusion equation (66) in terms of its Galerkin approximants of finite-dimension ([12]) which turns the Partial differential governing equations into ordinary differential stochastic equations

$$U^{(n)}(x,t) = \sum_{i=1}^{n} U_i^{(n)}(t)\varphi_i(x) \tag{83-a}$$

$$\eta^{(n)}(x,t) = \sum_{i=1}^{n} b_i^{(t)}\,\varphi_i(x) \tag{83-b}$$

$$(-\Delta\,\varphi_\ell)(x) = \lambda_\ell\,\varphi_\ell(x) \tag{83-c}$$

$$\begin{aligned} \frac{dU_i^{(n)}(t)}{dt} &= -\,(\lambda_i\,U_i^{(n)}(t)) \\ &\quad - \nabla V_i^{(n)}\,(U_0^{(n)}(t),\cdots,U_n^{(n)}(t)) \\ &\quad + \frac{db_i(t)}{dt} \end{aligned} \tag{83-d}$$

where the Brownian stirring have the correlation function

$$E_{\{b\}}(b_i(t)\, b_j(t')) = kT\delta_{ij} \min\{t, t'\} \tag{84}$$

and

$$-\nabla \cdot V_i^{(n)}(U_i^{(n)}(t)) \equiv \int_\Omega d^\nu x\, V(U^{(n)}(x,t)) \cdot \varphi^i(x) \tag{85}$$

Here the measurable function $V(a_1, \ldots, a_n)$ on $R^n$ is defined by the following relationship with the non-linearity $V(x)$

$$\nabla_{aj} V_i^{(n)}\,(a_1, \ldots, a_n) \equiv \int_\Omega d^\nu x\, V(\sum_{j=1}^n a_j\, \varphi_j(x))\varphi^i(x) \tag{86}$$

In order to write the equilibrium probability distribution associated to the Galerkin approximant system of ordinary differential equations as expressed by eq.(83-d) we follows previous studies ([11]) and write this equilibrium measure on $R^n$ directly from the structural form as given by the Langevin stochastic equation eq.(83-d). We obtain as a result the following finite - dimensional probability measure as a standard expected result ([11])

$$\begin{aligned}
&d^{eq}\mu_{(n)}[U_0^{(n)}, \ldots, U_i^{(n)}, \ldots U_n^{(n)}] = \\
&= \frac{1}{Z} \times \exp\left\{ -\frac{1}{kT} \sum_{j=1}^n [V_j^{(n)}(U_0^{(n)}, \cdots, U_j^{(n)}, \cdots, U_n^{(n)})] \right\} \\
&\times \exp\left\{ -\frac{1}{kT} \sum_{i=1}^n (\lambda_i (U_i^{(n)})^2 \right\} \\
&\times (\prod_{l=1}^n dU_i^{(n)})
\end{aligned} \tag{87}$$

here $Z$ is the associated probability distribution normalization factor and the set of measures $\{d^{eq}\mu_{(n)}\}$ forms a ordered chain of measures on the phase space $R^\infty = \prod_{i=1}^\infty R_i$, i.e: $d_n^{eq}\,\mu < d_{n+1}^{eq}\mu$ (see appendices for comments on this question).

The unique accumulation point of the set of weakly compact measures eq.(87) on $C(R^\infty, R)$ can be shown to exist under certain rigorous mathematical conditions (by a

direct application of the Prokhorov-Nguyen Zui Tien weak measure convergence criterium - which will not be fully used here).

However for those non-linearities $V(x)$ such that the associated "potentials" $\{V^{(n)}(a_1, \dots, a_n)\}$ make a set of positive cylindrical functions on $C(R^\infty, R)$, one can see straightforwardly that the associated "potential" term on eq.(87) for each $n$ is always bounded by the function 1, which is an integrable function on the phase space $R^\infty = \prod_{i=0}^{\infty} R$ equipped with the cylindrical Gaussian measure

$$
\begin{aligned}
d^{(0)}\,\mu[\{U_i\}]_{0<i<\infty} = \lim_{n\to\infty} & \left\{ \frac{1}{Z^{(0)}} \times \exp\left\{ -\frac{1}{kT} \sum_{i=1}^{n} \lambda_i (U_i^{(n)})^2 \right\} \right. \\
& \left. \times \left( \prod_{l=1}^{n} dU_i^{(n)} \right) \right\} \qquad (88)
\end{aligned}
$$

$$
= D^F[g(x)]\, e^{-\frac{1}{kT}\int_\Omega d^D x [g(Ag)](x)} \qquad (89)
$$

where we have re-writtten the "free" equilibrium measure on the physicist well-known notation.

Note that we have just applied the well-known basic Lebesgue convergence theorem of Real analysis to obtain the above claimed mathematical result on the rigorous existence of the lim sup of the set of measures as given by eq.(88).

It is very worth remark that in the case that $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} < \infty$, namely: $A_2^{-1}$ is a class trace operator like the Lapacean $(-\Delta)^\alpha + \mu^2$ on $L^2(R^\nu)$ for $\alpha > \frac{\nu}{2}$, one can easily apply the well-know Minlos theorem 3, section 3 on Cylindrical measures supports to "reduce" the previous equilibrium phase space [ which is a purely topological product space-without a priori vector structure]: $\prod_{i=0}^{\infty} R_i = R^\infty$ to the Hilbert-space $L^2(R^\nu)$ of the ensemble formed by the set of initial conditions $g(x)$ - eq.(67).

Let us now briefly comment on similar equilibrium functional-cylindrical measures for the anomalous diffusion [(11)] as written below in the notation with $V(x)$ being a positive square integrable function as supposed in our study and $\varepsilon$ a positive constant

$(\varepsilon = \int_{R^\nu} V^2(x)dx)$

$$\frac{\partial U(x,t)}{\partial t} = (-((-\Delta)^\alpha + \varepsilon) - V(x))U(x,t) + F(U(x,t)) + \eta(x,t) \tag{90}$$

$$U(x,0) = g(x) \tag{91}$$

Procceding as exposed above, we arrive straighforwardly at the rigorous mathematical defined equilibrium measure on $L^2(R^\nu)$ for the case of the positivity of the function $G(x) = \int_0^x F(\xi)\, d\xi\, in$ the non-linear anomalous diffusion problem eq.(83) - eq.(84).

$$d^{eq}\mu[g] = \exp\left\{-\frac{1}{kT}\int_{R^\nu} dx\, G(g(x))\right\} \times d^{(0)}_{((-\Delta)^\alpha+V+\varepsilon)}\,\mu[g] \tag{92}$$

Here $d^{(0)}_{((-\Delta^\alpha)+V+\varepsilon)}\,\mu[g]$ is the Gaussian cylindrical measure associated to the Generating functional defined by the free class trace operator $((-\Delta)^\alpha + v + \varepsilon)^{-1}$, since $((-\Delta)^{+\alpha} + V + \varepsilon)^{-1} = ((-\Delta)^{+\alpha} + \varepsilon)^{-1}\, V((-\Delta)^{+\alpha} + \varepsilon)^{-1}$ and $((-\Delta)^\alpha + \varepsilon)^{-1} \in \oint_1(R^\nu)$ for $\alpha > \frac{\nu}{2}$ with $V \in L^2(R^\nu)$ as it is supposed here

$$\begin{aligned} Z^{(0)}_\alpha\,[j(x)] &= \exp\left\{-\frac{1}{2}\int_{R^\nu} dx \int_{R^\nu} dy\, j(x)[(-\Delta)^\alpha + V + \varepsilon]^{-1}(x,j)\, j(y)\right\} \\ &= \int d^{(0)}_{((-\Delta^\alpha)+V+\varepsilon)}\,\mu[g]\ \exp\left\{i\int_{R^\nu} dx\, g(x)\, j(x)\right\} \end{aligned} \tag{93}$$

# 6 On white - noise destributional path integral representation.

In this short section of our study let us present the functional Fourier transform of the characteristic (Generating) functional associated to the often physical used white noise (generalized) process.

As a first step let us consider the following positive $S(R^\nu)$-continuous functional (with $\gamma \in R^+$)

$$Z_{\text{white}}[j] = \exp\left\{-\frac{1}{2}\gamma\langle j,j\rangle_{L^2(R^\nu)}\right\} \tag{94}$$

By a direct application of Minlos theorem, there is a cylindrical measure on $S'(R^\nu)$ (the tempered Schwartz distribution on $R^\nu$) representing eq.(94) in terms of a path integral

$$Z_{\text{white}}[j] = \int_{S'(R^\nu)} d\mu[T] \;\exp\, i(T(j)) \tag{95}$$

here $T$ is an tempered distribution integrated out on eq.(95) and $T(j)$ denotes its action on the element $j \in S(R^\nu)$.

At this point, we call the reader attention that one can not make a further reduction of the functional domain of eq.(95) to a space of functions as it was done in previous section in order to overcome the problem of "multiplying distributions".

Sometimes it is worth to represent the path-integral eq.(55) in the product space $R^\infty$ by considering the associated harmonic (Hermite) expansion ([11]) of the objects in the path integral representation

$$T = \lim\nolimits_{S'(R^\nu)} \left(\sum t_n\, H_n\right) \tag{96-a}$$

$$j = \lim\nolimits_{S(R^\nu)} \left(\sum j_n\, H_n\right) \tag{96-b}$$

In this harmonic expansion eq.(95), reads of as

$$\begin{aligned} Z_{\text{white}}[j_n] &= \exp\left\{-\frac{\gamma}{2}\sum_n j_n^2\right\} \\ &= \lim\nolimits_n \,\sup\left\{\int_{R^n} \frac{dt_0}{\sqrt{\pi\gamma}}\cdots\frac{dt_n}{\sqrt{\pi\gamma}}\, e^{-\frac{1}{2\gamma}\sum\limits_{\ell=0}^{n}(t_\ell)^2}\; e^{i(\sum\limits_{\ell=0}^{n} j_\ell\, t_\ell)}\right\} \end{aligned} \tag{97}$$

At this point one can give a straightforward proof of the result stated on the last reference of ref [10] obtained by complicated localization methods that for different values of the strenght white noise $\gamma_1$ and $\gamma_2$ $(\gamma, \neq \gamma_2)$ the associated path measures have different supports on the distributional space $S'(R^\nu)$.

This can be easily seen as a consequence of a direct application of the well-known Kakutani theorem ([4]) that says that if

$$\lim\sup_N \left\{\sum_{n=0}^{N}\left(\frac{\frac{1}{\gamma_1}}{\frac{1}{\gamma_2}} - \frac{\frac{1}{\gamma_2}}{\frac{1}{\gamma_1}}\right)^2\right\} = +\infty \tag{98}$$

the associated Gaussian measures are singular measure to each other (this support are disjoint functional measurable sets on $S'(R^\nu)!$ ). This result shows how different measure theory on infinite dimensional spaces are from the usual finite-dimensional ones.

# 7 On the invariant Ergodic path integral for a class of non-linear wave equation.

Let us start this last section of our studies by considering the discreticized ($N$- particle) wave motion Hamiltonian associated to a non-linear wave equation on a two-dimensional space-time

$$H(p_i, x_i) = \sum_{i=1}^{N} a\left(\frac{p_i^2}{2} + \frac{1}{2}(x_{i+1} - x_i)^2 + V(x_i)\right) \tag{99}$$

with the non-linearity given by a Lipschitz function $V(x)$ and the imposed initial and boundary conditions

$$x_i(-L) = x_i(L) = 0\ ;\ x_i(0) = x_i^{(0)}\ ;\ p_i(0) = p_i^{(0)} \tag{100}$$

Note that a means the lattice spacing of our "discreticized" string, namely at the limit of zero spacing eq.(99) is given by the continuum action

$$H[\pi(x,t)\,,\, U(x,t)] = \int_{-L}^{L} dx\left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial}{\partial x}U\right)^2 + V(U)\right)(x,t) \tag{101}$$

Since one can easily prove the existence and uniqueness of global solutions to the associated discreticized (and continuum) wave motion Hamiltonian equations for eq.(99), one can apply straightforwardly the Ergodic theorem to write the invariant measure for the Hamiltonian system ([11])

$$\lim_{T\to\infty}\int_0^T dt\, F[(x_i(t))] = \int_{R^N} F[(x_i)]\, d^{eq}\,\mu_{(N)}[(x_i)] \tag{102}$$

with the explicitly equilibrium measure (with $a = \frac{2L}{N}$ ).

$$d^{eq}_{(N)}\mu[(x_i)] = \frac{1}{Z}\,\exp\left[-\frac{1}{2kT}\left(\sum_{l=1}^{N} a\left(\frac{(x_{l+1} - x_l)^2}{a^2} + V(x_l)\right)\right)\right] \tag{103}$$

with $Z$ a (probabilistic) normalization factor and $kT$ the Boltzan parameters (temperature).

Let us remark that the set formed by these measures is a bounded set on the space $C(\dot{R}^\infty, R)$ (see appendix) and by the Alaoglu-Bourbaki theorem, it is a weakly compact set on the vague topology on its dual (formed by Baire measures).

As a consequence, one can extract a sub-sequence of the sequence of invariant ergodic measures given by eq.(103), such that its infinite-dimensional limit is well-defined

$$\begin{aligned}\lim_N \sup \left\{ \int_{\dot{R}^\infty} F(x^\infty) \cdot d^{eq}_{(N)}\mu[(x^i)] \right\} &= \\ = \int_{C^{\frac{1}{2}}([-L,L])} d^{Wienner}\, \mu[x(\sigma)] \times F(x(\sigma)) \times \\ \times \exp\left\{ -\frac{1}{2kT} \int_{-L}^{L} d\sigma\, V(x(\sigma)) \right\} \end{aligned} \tag{104}$$

here $d^{Wienner}\,\mu[x(\sigma)]$ is the path Wienner measure associated to the 1-dimensional Laplacean $-\frac{d^2}{d^2x}$ with Dirichlet Boundary conditions on the domain $[-L, L]$ (see eq.(74) - section 4) with the identification $x(\sigma) = U(x, 0)$.

We have, thus, our Ergodic theorem for the $1+1$ dimensional non-linear wave equation

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{T} \int_0^T F(U(x,t))dt \\ &= \int_{C^{\frac{1}{2}}([-L,L];x(-L)=x(L)=0)} d^{Wienner}\, \mu[x(\sigma)] \times F(x(\sigma)) \times e^{-\frac{1}{2kT}\int_{-L}^{L} d\sigma\, V(x(\sigma))} \end{aligned} \tag{105}$$

for any functional $F(x(\sigma))$ bolonging to the functional space $L^1\Big[C^{\frac{1}{2}}([-L, L];\, x(-L) = x(L) = 0);\, d^{Wienner}\,\mu[x(\sigma)]\Big]$.

# 8 On Polyakov's Bosonic String Path Integral - Revisited on the light of correct measures definition

In opinion of the author of ref. [13] "there are methods and formulae in science, which serve as master-key to many apparently different problems. The resources of such things

have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned sum over random paths. The replacement in necessary, because today gauge invariance plays the central role in physics" (A. M. Polyakov).

The general picture has been envisaged as follows [13]: one should try to solve loop-space or generalized Schrödinger functional wave equations by the appropriate flux lines functionals represented by transition amplitudes given by the sums over all possible surfaces with fixed boundary.

$$G(C) = \sum_{(S_C)} \exp\left\{-\frac{1}{2\pi\alpha'}\, A(S_c)\right\} \tag{106}$$

here $C$ is some loop (smooth or a random closed path), $S_C$ is a surface bounded by the loop $C$ and $A(S_C)$ is the area of this surface and $\alpha'$ an extrinsinc (lenght square) constant (the Regge slope parameter).

The main point on Polyakov's propose is to introduce besides the surface parametrization $X_\mu(\xi_1, \xi_2)$, an intrinsic metric tensor $g_{ab}(\xi_1, \xi_2)$ and a quadratic functional on the random surface $X_\mu(\xi_1, \xi_2)$ field substituting the area functional in eq. (106) (with $2\pi\alpha' = 1$)

$$A(S_C) = \frac{1}{2}\int_D d^2\xi(\sqrt{g}g^{ab}\partial_a\, X_\mu \partial_b X^\mu)(\xi) \tag{107}$$

It is very important to remark that the above 2D-gravity induced surface functional has the geometrical meaning of the area spanned by the surface $X_\mu(\xi_1, \xi_2)$ *only at the classical level* $\alpha' \to 0$ (see ref. [14] for a study for the pure geometrical Nambu-Goto action on the framework of these reparametrization invariant functional integrals).

In order to proceed to the quantum theory, A.M. Polyakov has proposed that the quantum surface average of any extended reparametrization invariant functional $\Phi[X_\mu(\xi_1, \xi_2); g_{ab}(\xi_1, \xi_2)]$ should be given by the following expression

$$\int d\mu[S]\phi(S_C) \stackrel{\text{def}}{\equiv} \int [Dg_{ab}(\xi)] \exp(-\mu_{bare}\int \sqrt{g}d^2\xi) \int [D[X_\mu(\xi)]$$
$$\left[\exp\left(-\frac{1}{2}\int_D (\sqrt{g}g^{ab}\partial_a X_\mu \partial_b X_\mu)(\xi)d^2\xi\right)\right] \Phi[X_\mu(\xi), g_{ab}(\xi)] \tag{108}$$

The reparametrization invariant functional measures on eq. (108) are associated to the following functional measures

$$\|\delta X^\mu\|^2 = \int d^2\xi (g(\xi))^{1/2}\, \delta X_\mu(\xi)\delta X_\mu(\xi) \tag{109}$$

and

$$\|\delta g_{ab}\|^2 = \int d^2\xi [g(\xi)]^{1/2}\, (g^{aa'}\, g^{bb'} + C g^{ab} g^{a'b'})\delta g_{ab}\delta g_{a'b'} \tag{110}$$

where $C \neq -\frac{1}{2}$ is an arbitrary constant.

The reparametrization invariant gaussian functional integral $X_\mu(\xi_1, \xi_2)$ is easily evaluated with the result in the conformal gauge $g_{ab} = \rho^2\delta_{ab}$ (for closed boundary-less 2D-compact Riemannian manifolds)

$$\det{}^{-D/2}(-\Delta_{g_{ab}=\rho^2\delta_{ab}}) = \exp\left\{\frac{D}{48\pi}\int d^2\xi \left[\frac{(\partial_a\rho)^2)}{\rho^2} + \left(\lim_{\varepsilon\to 0}\frac{D}{4\pi\varepsilon}\right)\rho^2\right]\right\} \tag{111}$$

The functional integration on the intrinsic metric field is well-known ([13]) with infinitesimal coordinate transformation $\{\in_a\ (\xi_1, \xi_2)\}$ around the conformal orbit (i.e., $\nabla^c_{g_{ab}=\rho^2\delta_{ab}} \cdot \in_c = 0$)

$$\|\delta g_{ab}\|^2 = (1+2c)\int d^2\xi\, \delta\rho(\xi)\delta\rho(\xi) + \int d^2\xi\, \sqrt{g}\phi^b_a\phi^a_b \tag{112}$$

Here

$$\phi_{ab} = (\nabla_a\ \in_b + \nabla_b\ \in_a)_{g_{ab}=\rho^2\delta_{ab}} \tag{113}$$

From eq. (112) we derive the correct integration measure in terms of the Feynman measures, denoted by the symbol $D^F(\cdot) = \prod_\xi d(\cdot)$

$$[Dg_{ab}(\xi)] = D^F[\rho(\xi)]D^F[\in_a\ (\xi)](det^{1/2}\mathcal{L}) \tag{114}$$

Here the Polyakov's operator $\mathcal{L}$ is obtained from eq. (112) and given by

$$(\mathcal{L}\ \in)_a = \nabla^b(\nabla_a\ \in_b + \nabla_b\ \in_a)|_{g_{ab}=\rho^2\delta_{ab}} \tag{115}$$

and its functional determinant was exactly evaluated (acting on smooth $C^\infty$ compact support vector-sections on $S$)

$$-\frac{1}{2}\log\det\mathcal{L} = \frac{13}{6\pi}\int_\xi \left( \left[\frac{1}{2}\frac{(\partial_a\rho)^2}{\rho^2}\right] + \int_\xi \left(\lim_{\varepsilon\to 0}\frac{2}{4\pi\varepsilon}\right)\rho^2(\xi)\right) d^2\xi \tag{116}$$

By combining eq. (111) with eq. (116) and eq. (108), we obtain the partition function for the closed surfaces *defined in terms of the natural conformal quantum degrees of freedom* $\rho(\xi_1,\xi_2)$

$$Z = \int D^F[\rho(\xi)] \exp\left(-\frac{(26-D)}{12\pi}\int_\xi \left[\frac{1}{2}\frac{(\partial_a\rho)^2}{\rho^2}\right] + \int_\xi \mu_R^2\rho^2\right) \tag{117}$$

This expression shows the origin of the commonly known critical dimension 26 in the string theory: at this value of the dimension one does not have dynamics for the metric field $g_{ab}(\xi) = \rho^2(\xi)\delta_{ab}$. However for $D < 26$ one must examine the "$\sigma$-model like" in eq. (117) which is not the Liouville field theory as originally stated by A.M. Polyakov [(13)] because the natural theory's dynamical variable in this framework is the scalar field $\rho(\xi)$ instead of that proposed initially by Polyakov $2lg\,\rho(\xi) = \varphi(\xi)$. These above cited 2D-theories coincides only for very weak fluctuations around the 2D-flat metric $\rho(\xi) = 1 + \varepsilon\overline{\rho}\,(\varepsilon \to 0)$ in our opinion.

Note that the quantum field equation associated to the obtained effective partition functional is given by (the *the two-dimensional* effective Einstein equations for this induced 2D-gravitation!)

$$(\partial^a\partial_a)\rho(\xi) = \frac{12\pi\mu_R^2}{(26-D)}(\rho(\xi))^3 + \frac{12\pi}{(26-D}\frac{(\partial_a\rho)^2}{\rho} \tag{118}$$

Note that our $\sigma$-model like (Euclidean) lagrangian (with $\mu_R^2 = \mu_{bare}^2 + \lim_{\epsilon\to 0^+}\frac{(2-D)}{4\pi\varepsilon}$) describing the closed random surface sum

$$\mathcal{L}(\rho,\partial_a\rho) = \frac{26-D}{12\pi}\int_\xi \left[\frac{1}{2}\partial_a\left(\frac{1}{\rho}\right)\partial_a(\rho)\right](\xi)d^2\xi + \mu_R^2\int_\xi \rho^2(\xi)d^2\xi \tag{119}$$

does not possesses in principle a full conformal symmetry as a consequence of the correct variable to be quantized. It is worth remark that even in the original Polyakov's work

the symmetry which remains after specification of the conformal gauge are the conformal transformation of the $\xi$-domain $|\frac{dw}{dz}|^2 = 1$ for $\phi(\,\overline{z})$ defined as a scalar field. We *conjecture* that the only phase in which the 2D-quantum field theory makes sense is its perturbative phase around the "flat" configuration $\rho^2(\xi) = 1 + \frac{1}{D}\rho_q^{-2}(\xi)$ in a $\frac{1}{D}$-expansion of other suitable classical $\rho_{cl}(\xi)$ solution of eq. (118) $\rho^2(\xi) = \rho_{cl}^2(\xi) + \frac{1}{D}\,\rho_q^{-2}(\xi)$.

The intercept point probabilities (the scalar $N$-scattering amplitude) in this random surface theory is straightforwardly reduced to the average

$$A^{(\delta)}(p^1, \cdots, p^N) = (\delta)^{(\sum_{i=1}^{N} p_i^2)} \int_\xi \Pi_{i=1}^N d^2\xi_j \, (\Pi_{i<j}^N |\xi_i - \xi_j|^{p^i \cdot p^j}) \\ \times \int D^F[\rho] e^{-\mathcal{L}(\rho, \partial_a \rho)} \, (\Pi_{i=1}^N [\rho(\xi_j)]^{+2(1-p_i^2)}) \tag{120}$$

It is possible to show that only for (Euclidean) values of external momenta $1 - p_i^2 = -1, -2, \cdots$ or $p_i^2 = 0, -1, -2, \cdots$, the quantum field average eq. (14) makes sense and suggesting, thus, to a spectrum without the usual lowest state being a tachyon.

So, our main conclusion is that the summation of Bosonic random surface understood as 2D-induced quantum gravitation as originally proposed by A.M. Polyakov in ref. [13] is reduced to a massive $\sigma$-model scalar field lagrangean obtained in eq. (117), and not to the Liouville somewhat ill-defined $2D$-quantum model as originaly put forward in ref. [13]. Note that the simplest supersymmetric version of the Bosonic Quantum Field eq. (117) describe the sum of fermionic random surfaces with critical dimension $D = 10$ and to be analyzed in the next section.

Let us finally point that there is a formal propose to describe the closed random surface partitional functional eq. (117) by means of Liouville- Poliakov degree of freedom $\phi(\xi) = 2lg\,\rho(\xi)$ which has the advantages of taking into account directly in the path integral *the positivity* of the quantum field $\rho(\xi)$. The important formal step in this study is the variable functional change

$$D^F[\rho(\xi)] = \Pi_\xi \, d[e^{\frac{\phi}{2}(\xi)}] = \Pi_\xi(\det(e^{\frac{\phi}{2}})(\xi)) \, d(\phi(\xi)) \tag{121}$$

Unfortunately the *functional Jacobian* $\det(e^{\frac{\phi}{2}})$ does not makes sense as a functional change of functional measures. However, one can propose a definition for the above cited Jacobian as in the original Fujikawa's "hand-wave" prescription to handle the axial anomaly as follows:

$$\begin{aligned}
&\det{}_F[(e^{\frac{\phi}{2}})(\xi)] = \lim_{\varepsilon\to 0^+} \exp Tr_{(\xi)}\, [lg(e^{\frac{\phi}{2}})(\xi)e^{-\varepsilon\Delta_{g_{ab}=e^{\phi}\delta_{ab}}}] = \\
&\lim_{\varepsilon\to 0^+} \exp\left\{\int d^2\xi\, e^{\phi(\xi)}\frac{\phi}{2}(\xi)\left[\frac{1}{4\pi\varepsilon} - \frac{1}{12\pi}(e^{-\phi}\Delta\phi)\right](\xi)\right\} = \\
&\exp\left\{\frac{1}{48\pi}\int_\xi \left[\frac{1}{2}(\partial_a\phi)^2\right]\right\} \exp\left\{\frac{1}{8\pi\varepsilon}\int_\xi e^{\phi(\xi)}\phi(\xi)\right\}
\end{aligned} \tag{122}$$

By analyzing eq. (122) we feel that eq. (122) is not sound as it stands since 1) one could use other regularizing operator as that one of eq. (115); 2) the term in front of kinetic term for the Liouville weight *decreases* and leading to a new (incorrect) critical dimension for string theory, etc... Anyway eq. (122) deserves further studies.

# 9 On Polyakov's Fermionic String Path Integral - Reivisited

In the last section of our paper we review the original paper by A.M. Polyakov (Quantum Geometry of Fermionic Strings (Phys. Lett. 103B, 211, 1981) ([15]) with *corrections and improvements* on the concepts exposed there and following as closely as possible to the original A.M. Polyakov's *paper writing*.

In this previous section 8, we have clarified and improved the Polyakov's procedure for quantizing Bosonic strings as 2D quantum gravity models by a carefull analysis of the involved path-integrals. It is, thus, very urgent to extend these results to fermionic strings because, as was shown in refs. [16] its Kaluza-Klein reduction to certain manifolds are expected to represent from a Q.F.T. point of view the $N = 4$ extended Super-Yang Mills Field Theory (The Maldecena suggestion).

Let us begin from the supersymmetric extension of the Bose string (quantum gravity!) lagrangean. ([15])

$$S = \frac{1}{2\pi\alpha'}\left\{\int d^2\xi\left[\frac{1}{2}\sqrt{g}g^{\alpha\beta}\partial_\alpha X^A\partial_\beta X_A + \frac{1}{2}\overline{\psi}^A(i\gamma^\alpha\partial_\alpha)\psi_A\right.\right.$$
$$\left.\left.\overline{\chi}_\alpha\gamma^\beta\gamma^\alpha(\partial_\beta\chi^A + \frac{1}{2}\chi_\beta\psi^A)\psi_A\right]\right\} + \mu\int_D d^2\xi\sqrt{g}(\xi) \tag{123}$$

Here, the surface is parametrized by $X_A = X_A(\xi)$, $(A = 1\cdots D)$; $\psi^A$ is a $\xi$-two component Majorana spinor, $g_{\alpha\beta}(\xi)$ is a metric tensor and $\chi_\alpha$ is a spinor gravitino field. The Polyakov's strategy as exposed in the previous paper, was to integrate out the $\chi^A$ and $\psi^A$ fields firstly and, then, he has examined the resulting theory of "induced $\xi$-supergravity". By choosing the "super-conformal" gauge

$$g_{\alpha\beta}(\xi) = \rho^2(\xi)\delta_{\alpha\beta}\,;\ \chi_\alpha(\xi) = (\gamma_\alpha\chi)(\xi) \tag{124}$$

Polyakov has showed that the only expression which satisfies all $\xi$-supersymmetries not destroyed by the super-conformed gauge eq. (124) is the direct supersymmetric extension of the Bosonic action written in ref. [13], eq. (2)

$$e^{-W} = \int D\psi^A DX_a e^{-S} \tag{125}$$

In terms of the original fields $\rho(\xi)$ and $\chi(\xi)$, the component form of eq. (125) can be (correctly) rewritten as (with $2\pi\alpha' = 1$)

$$W[\rho,\chi] = \frac{10-D}{8\pi}\int [\frac{1}{2}\left(\frac{\partial_\xi\rho}{\rho}\right) + \left[\frac{1}{2}i\overline{\chi}(\gamma\partial)\chi + \frac{1}{2}\mu(\overline{\chi}\gamma_5\chi)\rho + \frac{1}{2}\mu^2\rho^2\right](\xi)d^2\xi \tag{126}$$

Note that in the usual Liouville field parametrization the induced 2D-supergravity is written as ($\rho = e^{\varphi/2}$)

$$W[\varphi,\chi] = \frac{10-D}{8\pi}\int_\xi\left[\frac{1}{2}(\partial W)^2 + \frac{1}{2}i\overline{\chi}(\gamma\partial)\chi + \frac{1}{2}\mu(\overline{\chi}\gamma_5\chi)e^\varphi + \frac{1}{2}\mu^2 e^{2\varphi}\right](\xi) \tag{127}$$

At this point it is worth remark that the intrinsic fermionic degrees of freedom in eq. (127) may be easily integrated out with the following result: [if one considers $\chi(\xi)$ as an

usual 2D-Dirac fermion field] ([17])

$$\int D^F[\chi(\xi)D^F[\overline{\chi}(\xi)] \exp\left\{-\left(\frac{10-D}{8\pi}\right)\int_\xi d^2\xi\left[\frac{1}{2}i\overline{\chi}(\gamma\partial)\chi+\frac{1}{2}\mu(\overline{\chi}\gamma_5\chi)\rho\right]\right\}$$
$$= \det\left[i\gamma\partial+\frac{1}{2}\mu\gamma_5\rho\right] = I(\rho) \tag{128}$$

At this point, we note that (after introducing the notation $\sigma_+ = \beta\left(\frac{1+\gamma_5}{2}\right)\overline{\beta}$ and $\sigma_- = \beta\left(\frac{1-\gamma_5}{2}\right)\overline{\beta}$, we have the $\mu$-expansion ($\mu << 1$)

$$I(\rho) = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2}\mu)^n}{n!}\int d^2\xi_1\cdots d^2\xi_n$$
$$\int D\beta D\overline{\beta}\exp\left[-\frac{1}{2}\int_\xi(\overline{\beta}(i\gamma\partial)\beta)\right]$$
$$(\sigma_+\rho-\sigma_-\rho)(\xi_1)\cdots(\sigma_+\rho-\sigma_-\rho)(\xi_n) \tag{129}$$

and it is a result of a well-known theorem on 2D-Fermionic model's that the only non zero terms of eq. (128) are those with equal number of $\sigma_+^{`}s$ and $\sigma_-^{`}s$. We get, thus, that eq. (128) becomes the bosonized path-integral below written

$$I(\rho) = \int D^F[a(\xi)]\exp\left(-\frac{1}{2}\int d^2\xi(\partial a)^2(\xi)\right)\exp\left(-\int d^2\xi\left[\frac{1}{2}\mu\, e^{\Delta(0)}\frac{1}{4\pi}\,\mathrm{sen}(\sqrt{4\pi a}+\rho)\right](\xi)\right) \tag{130}$$

where the (bare) $\xi$-cosmological constant $\mu$ (gets a multiplicative ultraviolet) renormalization $\mu_R = \frac{1}{2}\mu(\varepsilon)^{-\frac{1}{2\pi}}$.

As a final comment let us use as dynamical degrees of freedom the Polyakov's original conformal factor $\varphi(\xi) = lg\,\rho(\xi)$. In terms of this variable the bosonized theory's path integral is written as

$$Z = \int D^F[e^{\varphi(\xi)}]\exp\left\{-\frac{1}{2}\int_\xi\left[(\partial\varphi)^2(\xi)+\mu^2\left(\frac{10-D}{8\pi}\right)e^{2\sqrt{\frac{8\pi}{10-D}}\varphi}\right](\xi)\right\}$$
$$\times\left\{\int D^F[a(\xi)]\exp\left(-\frac{1}{2}\int d^2\xi(\partial a)^2(\xi)\right)\right.$$
$$\left.\exp\left(-\int d^2\xi\left[\frac{1}{8\pi}\mu_R\sin\,(\sqrt{4\pi a})e^{\sqrt{\frac{8\pi}{10-D}}\varphi}\right](\xi)\right)\right\} \tag{131}$$

It is worth to note that the one must use as the Feynman product measure that written in eq. (130) $\Pi_\xi(e^{\varphi(\xi)}dW(\xi))$ since the associated functional ($\xi$-covariant) functional metric is given by

$$\|\delta g_{ab}\|^2 = \int_\xi (e^{2\varphi(\xi)}\delta\varphi \cdot \delta\varphi)(\xi) d^2\xi = \int_\xi [\delta(e^\varphi)\delta(e^\varphi)](\xi) d^2\xi \tag{132}$$

Note that only for weak intrinsic metric fluctuations (or for $D = 10 - \varepsilon$) $e^{\varphi(\xi)}$ may be replaced directly by $\varphi(\xi)$ inside the Feynman product measure as it was supposed in Polyakov's original paper.

# Appendix A

In this appendix we give new functional analytic proofs of the Bochner-Martin-Kolmogorov Theorem of section II.

Theorem of Bochner-Martin-Kolmogorov (Version I) let $f : E \to R$ be a given real function with domain being a vector space $E$ and satisfying the following properties

1) $f(0) = 1$

2) The restriction of $f$ to any finite-dimensional vector sub-space of $E$ is the Fourier Transform of a real continuous function of compact support.

Then there is a measure $d\mu(h)$ on a $\sigma$-algebra containing the Borelians if the Space of Linear Functionals of $E$ with the topology of pontual convergence denoted by $E^{alg}$ such that for any $y \in E$

$$f(g) = \int_{E^{alg}} \exp(ih(g))d\mu(h) \tag{A.1}$$

**Proof:** Let $\{\hat{e}_{\lambda \in A}\}$ be a Hamel (Vectorial) basis of $E$ and $E^{(N)}$ a given sub-space of $E$ of finite-dimensional. By the hypothesis of the Theorem, we have that the restriction of the functions to $E^{(N)}$ (generated by the elements of the Hamel basis $\{\hat{e}_{\lambda_1}, \dots, \hat{e}_{\lambda_N}\} = \{e_\lambda\}_{\lambda \in \Lambda_F}$ is given by the Fourier Transform

$$f\left(\sum_{\ell=1}^{N} \sigma\lambda_\ell \hat{e}\lambda_\ell\right) = \int_{\prod_{\lambda \in \Lambda_F} R^\lambda} (dP_{\lambda_1} \cdots dP_{\lambda_N}) \exp\left[\sum_{\ell=1}^{N} a_{\lambda_\ell} P_{\lambda_\ell}\right] \hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \tag{A.2}$$

with $\hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \in C_c\left(\prod_{\lambda \in \Lambda_F} R^\lambda\right)$

As a consequence of the above written result we consider the following well-defined family of linear positive functionals on the space of continuous function on the product space of the Alexandrov Compactifications of $R$ denoted by $R^w$ :

$$L_{\lambda_F} \in \left[C\left(\prod_{\lambda \in \Lambda_F} (R^w)^\lambda; R\right)\right]^{Dual} \tag{A.3}$$

with

$$L_{\Lambda_F}[\hat{g}(P_{\lambda_1},\ldots,P_{\lambda_N})] = \int_{\prod\limits_{\lambda\in\Lambda_F}(R^w)^\lambda}\int \hat{g}(P_{\lambda_1},\ldots,P_{\lambda_N})(dP_{\lambda_1}\ldots dP_{\lambda_N}) \tag{A.4}$$

Here $\hat{g}(P_{\lambda_1},\cdots,P_{\lambda_N})$ still denotes the unique extension of eq.(A-2) to the Alexandrov Compactification $R^w$.

We remark noe that the above family of linear continuous functionals have the following properties:

1) The norm of $L_{\lambda_F}$ is always the unity since

$$||L_{\lambda_1}|| = \int_{\prod\limits_{\lambda\in\Lambda_F}(R^w)^\lambda} \hat{g}(P_{\lambda_1},\ldots,P_{\lambda_N})dP_{\lambda_1}\ldots dP_{\lambda_N} = 1 \tag{A.5}$$

2) If the index set $\Lambda_F$, contains $\Lambda_F$ the restriction of the associated linear functional $\Lambda_F$, to the space $C\left(\prod\limits_{\lambda\in\Lambda_f}(R^w)^\lambda, R\right)$ coincides with $L_{\Lambda_F}$.

Now a simple application of the Stone-Weisrtress Theorem show us that the topological closure of the union of the sub-space of functions of finite variable is the space $C\left(\prod\limits_{\lambda\in A}(R^w)^\lambda, R\right)$, namely

$$\overline{\bigcup_{\Lambda_F\subset A} C\left(\prod_{\lambda\in\Lambda_F}(R^w)^\lambda, R\right)} = C(\prod_{\lambda\in A}(R^w)^\lambda, R) \tag{A.6}$$

where the union is taken over all family of sub-sets of finite elements of the index set $A$.

As a consequence of the remark 2 and eq.(A-6) there is a unique extension of the family of linear functionals $\{L_{\Lambda_F}\}$ to the whole space $C\left(\prod\limits_{\lambda\in A}(R^w)^\lambda, R\right)$ and denoted by $L_\infty$. The RieszMarkov Theorem give us a unique measure $d\overline{\mu}(h)$ on $\prod\limits_{\lambda\in A}(R^w)^\lambda$ representing the action of this functional on $C\left(\prod\limits_{\lambda\in A}(R^w)^\lambda, R\right)$.

We have, thus, the following functional integral representation for the function $f(g)$:

$$f(g) = \int_{(\prod\limits_{\lambda\in A}(R^w)^\lambda)} \exp(i\bar{h}(g))d\mu(\bar{h}) \tag{A.7}$$

Or equivalently (since $\overline{h}(g) = \sum_{i=1}^{N} p_i a_i$ for some $\{p_i\}_{i \in N} < \infty$), we have the result

$$f(g) = \int_{(\prod\limits_{\lambda \in A} R^\lambda)} (\exp ih(g)) d\mu(h) \tag{A.8}$$

which is the proposed theorem with $h \in (\prod\limits_{\lambda \in \lambda_F} R^\lambda)$ being the element which has a the image of $\overline{h}$ on the Alexandrov Compactification $\prod\limits_{\lambda \in \lambda_F} (R^w)^\lambda$.

The practical use of the Bochner-Martin Kolmogorov Theorem is difficulted by the present day non existence of an algorithm generating explicitly a Hamel (Vectorial) Basis on Function of Spaces. However, if one is able to apply the theorems of section III one can construct explicitly the functional measure by only considering Topological Basis as in the Gaussian Functional integral eq.(32).

**Theorem of Bochner-Martin-Kolmogorov (Version 2)**

We have now the same hypothesis and results of theorem version 1 but with the more general condition.

3) The restriction of $f$ to any finite-dimensional vector sub-space of $E$ is the Fourier Transform of a real continuous function vanishing at "infinite".

For the proof of the theorem under this more general mathematical condition, we will need two lemmas and some definitions.

**Definitions 1.** Let $X$ be a normal Space, locally compact and satisfying the following $\sigma$-compacity condition

$$X = \bigcup_{n=0}^{\infty} K_k \tag{A.9}$$

with

$$K_n \subset \mathrm{int}(K_{n+1}) \subset K_{n+1} \tag{A.10}$$

we define the following space of continuous function "vanishing" at infinite

$$\tilde{C}_0(X,R) = \left\{ f(x) \in C(X,R) | \lim_{n\to\infty} \sup_{x\in(K_n)^c} |f(x)| = 0 \right\} \tag{A.11}$$

We have, thus, the following lemma.

**Lemma 1.** The Topological closure of the functions of compact support contains $\tilde{C}_0(X,R)$ in the topology of uniform convergence.

**Proof:** Let $f(x) \in \tilde{C}_0(X,R)$ and $g_\mu \in C(X,R)$, the (Uryhson) functions associated to the closed disjoints sets $\overline{K}_n$ and $(K^c_{n+1})$. Now it is straightforwardly to see that $(f \cdot g_n)(x) \in C_i(X,R)$ and converges uniformly to $f(x)$ due to the definition (A-11).

At this point, we consider a linear positive continuous functional $L$ on $\tilde{C}_0(X,R)$. Since the restriction of $L$ to each sub-space $C(K_n,R)$ satisfy the conditions of the Riesz-Markov Theorem, there is a unique measure $\mu^{(n)}$ on $K_n$ containing the Borelians on $K_n$ and representing this linear functional restriction. We now use the hypothesis eq.(A-10) to have a well defined measure on a $\sigma$-algebra containing the Borelians of $X$

$$\bar{\mu}(A) = \limsup \mu^{(n)}(A \bigcap K_n) \tag{A.12}$$

for $A$ in this $\sigma$-algebra and representing the functional $L$ on $\tilde{C}_0(X,R)$

$$L(f) = \int_X f(x) d\bar{\mu}(x) \tag{A.13}$$

Note that the normally of the Topological Space $X$ is a fundamental hypothesis used in this proof by means of the Uryhson lemma.

Unfortunatelly, the non-countable product space $\prod_{\lambda\in A} R^\lambda$ is not a Normal Topological Space (the famous Stone counter example) and we can not, thus, apply the above lemma to our Vectorial case eq.(A-8). However, we can overcame the use of the Stone Weirstrass Theorem in the Proof of the Bochner-Martin-Kolmogorov Theorem by considering directly a certain Functional Space instead of that given by eq.(A-6).

We define, thus, the following Space of Infinite-Dimensional functions vanishing at finite

$$C_0(R^\infty, R) \equiv C_0\left(\prod_{\lambda\in A} R^\lambda, R\right) \stackrel{\text{def}}{=} \overline{\bigcup_{\Lambda_F\subset A} \tilde{C}_0\left(\prod_{\lambda\in\lambda_F} R^\lambda, R\right)} \tag{A.14}$$

where the closure is taken in the topology of uniform convergence.

If we consider a given continuous linear functional $L$ on $C_0\left(\prod_{\lambda\in A} R^\lambda, R\right)$ there is a unique measure $\mu^\infty$ on the union of the Borelians $\prod_{\lambda\in\Lambda_F} R^\lambda$ representing the action of $L$ on $C_0(R^\infty, R)$.

Conversely, given a family of consistent measures $\{\mu_{\Lambda_F}\}$ on the finite-dimensional spaces $(\prod_{\lambda\in\Lambda_F} R^\lambda)$ satisfying the property of $\mu_{\Lambda_F}\left(\prod_{\lambda\in\Lambda_F} R^\lambda\right) = 1$, there is a unique measure on the cylinders $\prod_{\lambda\in A} R^\lambda$ associated to the functional $L$ on $C_0\left(\prod_{\lambda\in A} R^\lambda, R\right)$.

Collecting the results of the above written lemmas we get the Proof of eq.(A-8) in this more general case.

# Appendix B

## On the support evaluations of Gaussian Measures

Let us show explicitly by one example of ours of the quite complex behavior of cylindrical measures on infinite dimensional spaces $R^\infty$.

Firstly we consider the family of Gaussian measures on $R^\infty = \{(x_n)_{1 \le n \le \infty}, x_n \in R\}$ with $\sigma_n \in \ell^2$.

$$d^{(\infty)}\mu(\{x_n\}) = \lim_N \sup \left\{ \prod_{n=1}^{N} (dx_n \frac{1}{\sqrt{\sigma_n \pi}}) e^{-\frac{x_n^2}{2\sigma_n^2}} \right\} \tag{B-1}$$

Let us introduce the measurable sets on $R^\infty$

$$E_{(\alpha_n)} = \left\{ (x_n) \in R^\infty \ ; \ \|x\|^2_{(x_n)} = \sum_{n=1}^{\infty} \alpha_n^2 \, x_n^2 < \infty \right\}$$

$$\text{and } \sum_{\ell=1}^{\infty} \alpha_n^2 \sigma_n^2 < \infty \tag{B-2}$$

Here $\{\alpha_n\}$ is a given sequence suppose to belonging to $\ell^2$ either.

Now it is straightforward to evaluate the "mass" of the infinite-dimensional set $E_{(x_n)}$, namely

$$\begin{aligned} {}^{(\infty)}\mu(E_{(\alpha_n)}) &= \int_{R^\infty} d^\infty \mu(\{x_n\}) \Big[ \lim_{\varepsilon \to 0^+} e^{-\varepsilon(\sum\limits_{n=1}^{\infty} \alpha_n^2 x_n^2)} \Big] \\ &= \lim_{\varepsilon \to 0^+} \left\{ \lim \sup_{0 \le \ell \le n} \Big[ \prod_{\ell=1}^{n} (1 + 2\varepsilon \, \alpha_n^2 \sigma_n^2)^{-\frac{1}{2}} \Big] \right\} \end{aligned} \tag{B-3}$$

Note that

$$\left( \prod_{\ell=1}^{n} (1 + 2\varepsilon \, \alpha_n^2 \sigma_n^2)^{-\frac{1}{2}} \right) \le \frac{1}{1 + \sum\limits_{\ell=1}^{n} \alpha_n^2 \sigma_n^2} \tag{B-4}$$

As a consequence one can exchange the order of the limits on eq.(B-3) and arriving at

the result

$$
\begin{aligned}
{}^{(\infty)}\mu(E_{(\alpha_n)}) &= \lim \sup_{0\le\ell\le n} \left\{ \lim_{\varepsilon\to 0^+} \left[ \prod_{\ell=1}^{n} (1+2\varepsilon\,\alpha_n^2\sigma_n^2)^{-\frac{1}{2}} \right] \right\} \\
&= \lim \sup_{0\le\ell\le n} \{1\} = 1 \qquad \text{(B-5)}
\end{aligned}
$$

So we conclude on basis if eq.(B-5) that the support of the measure eq.(B-1) is the set $E_{(\alpha_n)}$ for any possible sequence $\{\alpha_n\} \in \ell^2$. Let us show that $(E_{(\alpha_n)})^C \cap E_{(\beta_n)} \neq \{\phi\}$, so these sets are not coincident.

Let be the sequences

$$
\begin{aligned}
\sigma_n &= n^{-\sigma} \\
\alpha_n &= n^{\sigma-1} \\
\beta_n &= n^{\sigma-\lambda} \qquad \text{(B-6)}
\end{aligned}
$$

with $\gamma > 1$ and $\sigma > 0$.

We have that

$$
\sum \alpha_n^2\sigma_n^2 = \sum \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(B-7)}
$$

$$
\sum \beta_n^2\sigma_n^2 = \sum n^{-2\lambda} < \infty \qquad \text{(B-8)}
$$

So $E_{\{\alpha_n\}}$ and $E_{\{\beta_n\}}$ are non-empty sets on $R^\infty$.

Let us consider the point $\{\bar{x}_n\} \in R^\infty$ and defined by the relationship

$$
\bar{x}_n^2 = n^{-2(\sigma-1)-\varepsilon} \qquad \text{(B-9)}
$$

We have that

$$
\begin{aligned}
\sum (\bar{x}_n)^2\alpha_n^2 &= \sum n^{-2(\sigma-1)-\varepsilon} \cdot n^{n^2(\sigma-1)} \\
&= \sum n^{-\varepsilon}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum (\bar{x}_n)^2\,\beta_n^2 &= \sum n^{-2(\sigma-1)-\varepsilon} \cdot n^{2(\sigma-\lambda)} \\
&= \sum n^{2-\varepsilon-2\lambda}
\end{aligned}
$$

If we choose $\varepsilon = 1$ ; $\gamma > 1$ ($\gamma = \frac{3}{2}!$), we obtain that the point $\{\bar{x}_n\}$ belongs to the set $E_{\{\beta_n\}}$ (since $\sum n^2 = \frac{\pi^2}{6}$), however it does not belongs to $E_{\{x_n\}}$ (since $\sum_{n=0}^{\infty} n^{-1} = +\infty$ ), although the support of the measure eq.(B-1) is any set of the form $E_{\{\gamma_n\}}$ with $\{\gamma_n\} \in \ell^2$.

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